

EQUIVARIANT SEMIPROJECTIVITY

N. CHRISTOPHER PHILLIPS

ABSTRACT. We define equivariant semiprojectivity for C^* -algebras equipped with actions of compact groups. We prove that the following examples are equivariantly semiprojective:

- Arbitrary finite dimensional C^* -algebras with arbitrary actions of compact groups.
- The Cuntz algebras \mathcal{O}_d and extended Cuntz algebras E_d , for finite d , with quasifree actions of compact groups.
- The Cuntz algebra \mathcal{O}_∞ with any quasifree action of a finite group.

For actions of finite groups, we prove that equivariant semiprojectivity is equivalent to a form of equivariant stability of generators and relations. We also prove that if G is finite, then $C^*(G)$ is graded semiprojective.

Semiprojectivity has become recognized as the “right” way to formulate many approximation results in C^* -algebras. The standard reference is Loring’s book [20]. The formal definition and its basic properties are in Chapter 14 of [20], but much of the book is really about variations on semiprojectivity. Also see the more recent survey article [5]. There has been considerable work since then.

In this paper, we introduce an equivariant version of semiprojectivity for C^* -algebras with actions of compact groups. (The definition makes sense for actions of arbitrary groups, but seems likely to be interesting only when the group is compact.) The motivation for the definition and our choice of results lies in applications which will be presented elsewhere. We prove that arbitrary actions of compact groups on finite dimensional C^* -algebras are equivariantly semiprojective, that quasifree actions of compact groups on the Cuntz algebras \mathcal{O}_d and the extended Cuntz algebras E_d , for finite d , are equivariantly semiprojective, and that quasifree actions of finite groups on \mathcal{O}_∞ are equivariantly semiprojective. We also give, for finite group actions, an equivalent condition for equivariant semiprojectivity in terms of equivariant stability of generators and relations.

In a separate paper [26], we prove the following results relating equivariant semiprojectivity and ordinary semiprojectivity. If G is finite and (G, A, α) is equivariantly semiprojective, then $C^*(G, A, \alpha)$ is semiprojective. If G is compact and second countable, A is separable, and (G, A, α) is equivariantly semiprojective, then A is semiprojective. Examples show that finiteness of G is necessary in the first statement, and that neither result has a converse.

We do not address equivariant semiprojectivity of actions on Cuntz-Krieger algebras, on $C([0, 1]) \otimes M_n$, $C(S^1) \otimes M_n$, or dimension drop intervals (except for a result for $C(S^1)$ which comes out of our work on quasifree actions; see Remark 3.14), or

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on $C^*(F_n)$. We presume that suitable actions on these algebras are equivariantly semiprojective, but we leave investigation of them for future work.

We also presume that there are interesting and useful equivariant analogs of weak stability of relations (Definition 4.1.1 of [20]), weak semiprojectivity (Definition 4.1.3 of [20]), projectivity (Definition 10.1.1 of [20]), and liftability of relations (Definition 8.1.1 of [20]). Again, we do not treat them. (Equivariant projectivity will be discussed in [26].)

Finally, we point out work in the commutative case. It is well known that $C(X)$ is semiprojective in the category of commutative C^* -algebras if and only if X is an absolute neighborhood retract. Equivariant absolute neighborhood retracts have a significant literature; as just three examples, we refer to the papers [15], [3], and [2]. (I am grateful to Adam P. W. Sørensen for calling my attention to the existence of this work.)

This paper is organized as follows. Section 1 contains the definition of equivariant semiprojectivity, some related definitions, and the proofs of some basic results.

Section 2 contains the proof that any action of a compact group on a finite dimensional C^* -algebra is equivariantly semiprojective. As far as we can tell, traditional functional calculus methods (a staple of [20]) are of little use here. We use instead an iterative method for showing that approximate homomorphisms from compact groups are close to true homomorphisms. For a compact group G , we also prove that equivariant semiprojectivity is preserved when tensoring with any finite dimensional C^* -algebra with any action of G .

In Section 3, we prove that quasifree actions of compact groups on the Cuntz algebra \mathcal{O}_d and the extended Cuntz algebras E_d , for d finite, are equivariantly semiprojective. We use an iterative method similar to that used for actions of finite dimensional C^* -algebras, but this time applied to cocycles. Section 4 extends the result to quasifree actions on \mathcal{O}_∞ , but only for finite groups. The method is that of Blackadar [5], but a considerable amount of work needs to be done to set this up. We do not know whether the result extends to quasifree actions of general compact groups on \mathcal{O}_∞ .

In Section 5, we show that the universal C^* -algebra given by a bounded finite equivariant set of generators and relations is equivariantly semiprojective if and only if the relations are equivariantly stable. This is the result which enables most of the current applications of equivariant semiprojectivity. It is important for these applications that an approximate representation is only required to be approximately equivariant. We give one application here: we show that in the Rokhlin and tracial Rokhlin properties for an action of a finite group, one can require that the Rokhlin projections be exactly permuted by the group.

Section 6 contains a proof that for a finite group G , the algebra $C^*(G)$, with its natural G -grading, is graded semiprojective. This result uses the same machinery as the proof that actions on finite dimensional C^* -algebras are equivariantly semiprojective. We do not go further in this direction, but this result suggests that there is a much more general theory, perhaps of equivariant semiprojectivity for actions of finite dimensional quantum groups.

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1. DEFINITIONS AND BASIC RESULTS

The following definition is the analog of Definition 14.1.3 of [20].

Definition 1.1. Let G be a topological group, and let (G, A, α) be a unital G -algebra. We say that (G, A, α) is *equivariantly semiprojective* if whenever (G, C, γ) is a unital G -algebra, $J_0 \subset J_1 \subset \cdots$ are G -invariant ideals in C , $J = \overline{\bigcup_{n=0}^{\infty} J_n}$,

$$\kappa: C \rightarrow C/J, \quad \kappa_n: C \rightarrow C/J_n, \quad \text{and} \quad \pi_n: C/J_n \rightarrow C/J$$

are the quotient maps, and $\varphi: A \rightarrow C/J$ is a unital equivariant homomorphism, then there exist n and a unital equivariant homomorphism $\psi: A \rightarrow C/J_n$ such that $\pi_n \circ \psi = \varphi$.

When no confusion can arise, we say that A is equivariantly semiprojective, or that α is equivariantly semiprojective.

Here is the diagram:

$$\begin{array}{ccc}
 & C & \\
 & \downarrow \kappa_n & \\
 & C/J_n & \\
 \psi \nearrow & \downarrow \pi_n & \searrow \kappa \\
 A & \xrightarrow{\varphi} & C/J
 \end{array}$$

The solid arrows are given, and n and ψ are supposed to exist which make the diagram commute.

We suppose that Definition 1.1 is probably only interesting when G is compact. Blackadar has shown that, in the nonunital category, the trivial action of \mathbb{Z} on \mathbb{C} is not equivariantly semiprojective [6]. This is equivalent to saying the the trivial action of \mathbb{Z} on $\mathbb{C} \oplus \mathbb{C}$ is not equivariantly semiprojective in the sense defined here. In the unital category, the trivial action of any group on \mathbb{C} is equivariantly semiprojective for trivial reasons, but there are no other known examples of equivariantly semiprojective actions of noncompact groups.

We will also need the following form of equivariant semiprojectivity for homomorphisms. Our definition is *not* an analog of the definition of semiprojectivity for homomorphisms given before Lemma 14.1.5 of [20]. Rather, it is related to the second step in the idea of two step lifting as in Definition 8.1.6 of [20], with the caveat that lifting as there corresponds to projectivity rather than semiprojectivity of a C^* -algebra. It is the equivariant version of a special case of conditional semiprojectivity as in Definition 5.11 of [8].

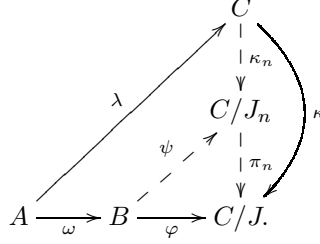
Definition 1.2. Let G be a topological group, let (G, A, α) and (G, B, β) be unital G -algebras, and let $\omega: A \rightarrow B$ be a unital equivariant homomorphism. We say that ω is *equivariantly conditionally semiprojective* if whenever (G, C, γ) is a unital G -algebra, $J_0 \subset J_1 \subset \cdots$ are G -invariant ideals in C , $J = \overline{\bigcup_{n=0}^{\infty} J_n}$,

$$\kappa: C \rightarrow C/J, \quad \kappa_n: C \rightarrow C/J_n, \quad \text{and} \quad \pi_n: C/J_n \rightarrow C/J$$

are the quotient maps, and $\lambda: A \rightarrow C$ and $\varphi: B \rightarrow C/J$ are unital equivariant homomorphisms such that $\kappa \circ \lambda = \varphi \circ \omega$, then there exist n and a unital equivariant homomorphism $\psi: B \rightarrow C/J_n$ such that

$$\pi_n \circ \psi = \varphi \quad \text{and} \quad \kappa_n \circ \lambda = \psi \circ \omega.$$

Here is the diagram:



The part of the diagram with the solid arrows is assumed to commute, and n and ψ are supposed to exist which make the whole diagram commute.

Remark 1.3. (1) Definition 1.1 is stated for the category of unital G -algebras.

Without the group, a unital C^* -algebra is semiprojective in the unital category if and only if it is semiprojective in the nonunital category. (See Lemma 14.1.6 of [20].) The same is surely true here, and should be essentially immediate from what we do, but we don't need it and do not give a proof.

- (2) In the situations of Definition 1.1 and Definition 1.2, we say that ψ *equivariantly lifts* φ .
- (3) In proofs, we will adopt the standard notation $\pi_{n,m}: C/J_m \rightarrow C/J_n$, for $m, n \in \mathbb{Z}_{>0}$ with $n \geq m$, for the maps between the different quotients implicit in Definition 1.1 and Definition 1.2. Thus $\pi_n \circ \pi_{n,m} = \pi_m$ and $\pi_{n,m} \circ \pi_{m,l} = \pi_{n,l}$ for suitable choices of indices. We further let $\gamma^{(n)}: G \rightarrow \text{Aut}(C/J_n)$ and $\gamma^{(\infty)}: G \rightarrow \text{Aut}(C/J)$ be the induced actions on the quotients.

Lemma 1.4. Let G be a topological group, let (G, B, β) be a unital G -algebra, let $A \subset B$ be a unital G -invariant subalgebra, and let $\omega: A \rightarrow B$ be the inclusion. If A is equivariantly semiprojective and ω is equivariantly conditionally semiprojective, then B is equivariantly semiprojective.

Proof. Let the notation be as in Definition 1.1 and Remark 1.3(3). Suppose $\varphi: B \rightarrow C/J$ is an equivariant unital homomorphism. Then equivariant semiprojectivity of A implies that there are n_0 and an equivariant unital homomorphism $\lambda: A \rightarrow C/J_{n_0}$ such that $\pi_{n_0} \circ \lambda = \varphi|_A$. Now apply equivariant conditional semiprojectivity of ω , with C/J_{n_0} in place of C and the ideals J_n/J_{n_0} , for $n \geq n_0$, in place of the ideals J_n . We obtain $n \geq n_0$ and an equivariant unital homomorphism $\psi: B \rightarrow C/J_n$ such that $\pi_n \circ \psi = \varphi$ (and also $\pi_{n,n_0} \circ \lambda = \psi|_A$). \square

Notation 1.5. Let (G, A, α) be a G -algebra. We denote by A^G the fixed point algebra

$$A^G = \{a \in A: \alpha_g(a) = a \text{ for all } g \in G\}.$$

In case of ambiguity of the action, we write A^α .

Further, if (G, B, β) is another G -algebra and $\varphi: A \rightarrow B$ is an equivariant homomorphism, then φ induces a homomorphism from A^G to B^G , which we denote by φ^G .

We need the following two easy lemmas.

Lemma 1.6. Let G be a compact group, and let (G, C, γ) be a G -algebra. Let $J \subset C$ be a G -invariant ideal. Then the obvious map $\rho: C^G/J^G \rightarrow C/J$ is injective and has range exactly $(C/J)^G$.

Proof. Injectivity is immediate from the relation $J \cap C^G = J^G$. It is obvious that $\rho(C^G/J^G) \subset (C/J)^G$. For the reverse inclusion, let $x \in (C/J)^G$. Let $\pi: C \rightarrow C/J$ be the quotient map. Choose $c \in C$ such that $\pi(c) = x$. Let μ be Haar measure on G , normalized so that $\mu(G) = 1$. Set

$$a = \int_G \gamma_g(c) d\mu(g).$$

Then $a \in C^G$ and $\pi(a) = x$. Therefore $a + J^G \in C^G/J^G$ and $\rho(a + J^G) = x$. \square

Lemma 1.7. Let G be a compact group, and let (G, A, α) be a G -algebra. Let $A_0 \subset A_1 \subset \cdots$ be an increasing sequence of G -invariant subalgebras of A such that $\overline{\bigcup_{n=0}^\infty A_n} = A$. Then $A^G = \overline{\bigcup_{n=0}^\infty A_n^G}$.

Proof. It is clear that $\overline{\bigcup_{n=0}^\infty A_n^G} \subset A^G$. For the reverse inclusion, let $a \in A^G$ and let $\varepsilon > 0$. Choose n and $x \in A_n$ such that $\|x - a\| < \varepsilon$. Let μ be Haar measure on G , normalized such that $\mu(G) = 1$. Then $b = \int_G \gamma_g(x) d\mu(g)$ is in A_n^G and satisfies $\|b - a\| < \varepsilon$. \square

Now we are ready to prove equivariant semiprojectivity of some G -algebras.

Lemma 1.8. Let G be a compact group, let $N \subset G$ be a closed normal subgroup, and let $\rho: G \rightarrow G/N$ be the quotient map. Let A be a unital C^* -algebra, and let $\alpha: G/N \rightarrow \text{Aut}(A)$ be an equivariantly semiprojective action of G/N on A . Then $(G, A, \alpha \circ \rho)$ is equivariantly semiprojective.

Proof. We claim that there is an action $\bar{\gamma}: G/N \rightarrow \text{Aut}(C^N)$ such that for $g \in G$ and $c \in C^N$ we have $\bar{\gamma}_{gN}(c) = \gamma_g(c)$. One only needs to check that $\bar{\gamma}$ is well defined, which is easy.

Let the notation be as in Definition 1.1 and Remark 1.3(3). Then $\varphi(A) \subset (C/J)^N$, which by Lemma 1.6 is the same as C^N/J^N . Let $\varphi_0: A \rightarrow C^N/J^N$ be the corestriction. Lemma 1.7 implies $J^N = \overline{\bigcup_{n=0}^\infty J_n^N}$, so semiprojectivity of $(G/N, A, \alpha)$ provides n and a unital G/N -equivariant homomorphism $\psi_0: A \rightarrow C^N/J_n^N$ which lifts φ_0 . We take ψ to be the following composition, in which the middle map comes from Lemma 1.6 and the last map is the inclusion:

$$A \xrightarrow{\psi_0} C^N/J_n^N \longrightarrow (C/J_n)^N \longrightarrow C/J_n.$$

Then ψ is G -equivariant and lifts φ . \square

Corollary 1.9. Let G be a compact group, let A be a unital C^* -algebra, and let $\iota: G \rightarrow \text{Aut}(A)$ be the trivial action of G on A . If A is semiprojective, then (G, A, ι) is equivariantly semiprojective.

Proof. In Lemma 1.8, take $N = G$. \square

Corollary 1.10. Let G be a compact group, and let (G, A, α) be a unital G -algebra. Then A is equivariantly semiprojective if and only if the inclusion of $\mathbb{C} \cdot 1$ in A is equivariantly conditionally semiprojective in the sense of Definition 1.2.

Proof. The subalgebra $\mathbb{C} \cdot 1$ is equivariantly semiprojective by Corollary 1.9, so we may apply Lemma 1.4. \square

Proposition 1.11. Let G be a compact group, and let $((G, A_k, \alpha^{(k)}))_{k=1}^m$ be a finite collection of equivariantly semiprojective unital G -algebras. Suppose that $l \in \{0, 1, \dots, m-1\}$. Set $A = \left(\bigoplus_{k=1}^l A_k\right) \oplus \mathbb{C}$ and set $B = \bigoplus_{k=1}^m A_k$, with the obvious direct sum actions $\alpha: G \rightarrow \text{Aut}(A)$ (with G acting trivially on \mathbb{C}) and $\beta: G \rightarrow \text{Aut}(B)$. Define $\omega: A \rightarrow B$ by

$$\omega(a_1, a_2, \dots, a_l, \lambda) = (a_1, a_2, \dots, a_l, \lambda \cdot 1_{A_{l+1}}, \lambda \cdot 1_{A_{l+2}}, \dots, \lambda \cdot 1_{A_m})$$

for

$$a_1 \in A_1, \quad a_2 \in A_2, \quad \dots, \quad a_l \in A_l, \quad \text{and} \quad \lambda \in \mathbb{C}.$$

Then ω is equivariantly conditionally semiprojective.

Proof. Let the notation be as in Definition 1.2 and Remark 1.3(3). For $k = 1, 2, \dots, l$ let $e_k \in A$ be the identity of the summand $A_k \subset A$, and for $k = 1, 2, \dots, m$ let $f_k \in B$ be the identity of the summand $A_k \subset B$. Set $q = 1 - \sum_{k=1}^l \mu(e_k)$. Let $P \subset B$ be the subalgebra generated by $f_{l+1}, f_{l+2}, \dots, f_m$. Then P is semiprojective and G acts trivially on it. Therefore Corollary 1.9 provides n_0 and a unital equivariant homomorphism $\psi_0: P \rightarrow qCq/qJ_{n_0}q$ such that $\pi_{n_0} \circ \psi_0 = \varphi|_P$. For $k = l+1, l+2, \dots, m$, set $p_k = \psi_0(e_k)$. Use equivariant semiprojectivity of A_k , with $p_k(C/J_{n_0})p_k$ in place of C and with $p_k(J_n/J_{n_0})p_k$ in place of J_n (for $n \geq n_0$) to find $n_k \geq n_0$ and a unital equivariant lifting

$$\psi_k: A_k \rightarrow \pi_{n_k, n_0}(p_k)(C/J_{n_k})\pi_{n_k, n_0}(p_k)$$

of $\varphi|_{A_k}$. Define $n = \max(n_1, n_2, \dots, n_m)$, and define $\psi: A \rightarrow C/J_n$ by

$$\psi(a_1, a_2, \dots, a_m) = (\kappa_n \circ \mu)(a_1, a_2, \dots, a_l) + \sum_{k=l+1}^m \pi_{n, n_k}(\psi_k(a_k)).$$

Then ψ is an equivariant lifting of φ . \square

Corollary 1.12. Let G be a compact group, and let $((G, A_k, \alpha^{(k)}))_{k=1}^m$ be a finite collection of equivariantly semiprojective unital G -algebras. Then $A = \bigoplus_{k=1}^m A_k$, with the direct sum action $\alpha: G \rightarrow \text{Aut}(A)$, is equivariantly semiprojective.

Proof. Proposition 1.11 (with $l = 0$) implies that the unital inclusion of \mathbb{C} in A is equivariantly conditionally semiprojective, so Corollary 1.10 implies that A is equivariantly semiprojective. \square

We can use traditional methods to give an example of a nontrivial action which is equivariantly semiprojective. This result will be superseded in Theorem 2.6 below, using more complicated methods, so the proof here will be sketchy.

Proposition 1.13. Let G be a finite cyclic group. Let G act on $C(G)$ by the translation action, $\tau_g(a)(h) = a(g^{-1}h)$ for $g, h \in G$ and $a \in C(G)$. Then $(G, C(G), \tau)$ is equivariantly semiprojective.

Proof. Let the notation be as in Definition 1.1 and Remark 1.3(3).

Take $G = \mathbb{Z}/d\mathbb{Z} = \{1, e^{2\pi i/d}, e^{4\pi i/d}, \dots, e^{2(d-1)\pi i/d}\} \subset S^1$. Let u be the inclusion of G in S^1 , which we regard as a unitary in $C(G)$. Then u generates $C(G)$ and $\tau_\lambda(u) = \lambda^{-1}u$ for $\lambda \in G$. Therefore it suffices to find n and a unitary $z \in C/J_n$ such that $\pi_n(z) = \varphi(u)$, $\text{sp}(z) \subset G$, and $\gamma_\lambda^{(n)}(z) = \lambda^{-1}z$ for all $\lambda \in G$.

Since $C(G)$ is semiprojective (in the nonequivariant sense), there are n_0 and a unitary $v_0 \in C/J_{n_0}$ such that $\pi_n(v_0) = \varphi(u)$ and $v_0^d = 1$. Moreover, for all $\lambda \in G$, we have

$$\lim_{n \rightarrow \infty} \|\pi_{n,n_0}(\gamma_\lambda^{(n_0)}(v_0)) - \lambda^{-1}v_0\| = 0.$$

Choose $\varepsilon > 0$ such that $\varepsilon < \frac{1}{2}|1 - e^{\pi i/d}|$, and such that whenever B is a unital C^* -algebra and $b \in B$ satisfies $\|b - 1\| < \varepsilon$, then $\|b(b^*b)^{-1/2} - 1\| < \frac{1}{2}|1 - e^{\pi i/d}|$. Choose n so large that $v = \pi_{n,n_0}(v_0)$ satisfies $\|\gamma_\lambda^{(n)}(v) - \lambda^{-1}v\| < \varepsilon$ for all $\lambda \in G$. Define $a \in C/J_n$ by

$$a = \frac{1}{d} \sum_{\lambda \in G} \lambda \gamma_\lambda^{(n)}(v).$$

Then one checks that $\gamma_\lambda^{(n)}(a) = \lambda^{-1}a$ for all $\lambda \in G$ and that $\|a - v\| < \varepsilon < 1$, so a is invertible. Set $w = a(a^*a)^{-1/2}$, and check that $\gamma_\lambda^{(n)}(w) = \lambda^{-1}w$ for all $\lambda \in G$. A calculation, using the choice of ε , shows that $\|w - v\| < \frac{1}{2}|1 - e^{\pi i/d}|$. So $e^{\pi i/d}G \cap \text{sp}(w) = \emptyset$. Let $f: S^1 \setminus e^{\pi i/d}G \rightarrow S^1$ be the function determined by $g(e^{it}) = e^{2\pi i k/d}$ when $t \in (\frac{2k-1}{d}, \frac{2k+1}{d})$. Then $f(\lambda\zeta) = \lambda f(\zeta)$ for all $\lambda \in G$ and $\zeta \in S^1 \setminus e^{\pi i/d}G$, and f is continuous on $\text{sp}(w)$. Define $z = f(w)$. The verification that z satisfies the required conditions is a calculation. \square

2. EQUIVARIANT SEMIPROJECTIVITY OF FINITE DIMENSIONAL C^* -ALGEBRAS

The main result of this section is that actions of compact groups on finite dimensional C^* -algebras are equivariantly semiprojective.

The main technical tool is a method for replacing approximate homomorphisms to unitary groups by nearby exact homomorphisms, in such a way as to preserve properties such as being equivariant. (In Section 6, we will also need to preserve the property of being graded.) The method used here has been discovered twice before, in Theorem 3.8 of [13] (most of the work is in Section 4 of [12], but the result in [12] uses the wrong metric on the groups) and in Theorem 1 of [18]. It is not clear from either of these proofs that the additional properties we need are preserved. We will instead follow the proofs of Theorem 5.13 and Proposition 5.14 of [1]. (We are grateful to Ilijas Farah for pointing out these references.)

Notation 2.1. For a unital C^* -algebra A , we let $U(A)$ denote the unitary group of A .

The following lemmas give an estimate whose proof is omitted in [1]. We will need this estimate again, in the proof of Lemma 3.9 below. (We don't get quite the same estimate as implied in [1].)

Lemma 2.2. Let Γ be a compact group with normalized Haar measure μ . Let A be a unital C^* -algebra. Suppose $r \in [0, \frac{1}{2}]$, and let $u: \Gamma \rightarrow U(A)$ be a continuous function such that $\|u(g) - 1\| \leq r$ for all $g \in G$. Then

$$\left\| \int_{\Gamma} u(g) d\mu(g) - \exp \left(\int_{\Gamma} \log(u(g)) d\mu(g) \right) \right\| \leq \frac{5r^2}{2(1-2r)}$$

and

$$\left\| \int_{\Gamma} u(g) d\mu(g) \right\| \leq 1.$$

Proof. The second statement is obvious.

For the first, we require the following estimates (compare with Lemma 5.15 of [1]): for $u \in U(A)$ with $\|u - 1\| < 1$, we have

$$(2.1) \quad \|\log(u) - (u - 1)\| \leq \frac{\|u - 1\|^2}{2(1 - \|u - 1\|)},$$

and for $a \in A$ with $\|a\| < 1$, we have

$$(2.2) \quad \|\exp(a) - (1 + a)\| \leq \frac{\|a\|^2}{2(1 - \|a\|)}.$$

Both are obtained from power series:

$$\|\log(u) - (u - 1)\| \leq \sum_{n=2}^{\infty} \frac{\|u - 1\|^n}{n} \leq \frac{1}{2} \sum_{n=2}^{\infty} \|u - 1\|^n$$

and

$$\|\exp(a) - (1 + a)\| \leq \sum_{n=2}^{\infty} \frac{\|a\|^n}{n!} \leq \frac{1}{2} \sum_{n=2}^{\infty} \|a\|^n.$$

Apply (2.1) to the condition $\|u(g) - 1\| \leq r$ and integrate, getting

$$(2.3) \quad \left\| \int_{\Gamma} \log(u(g)) d\mu(g) - \int_{\Gamma} u(g) d\mu(g) - 1 \right\| \leq \frac{r^2}{2(1 - r)}.$$

Since $r \leq \frac{1}{2}$, we also get

$$\left\| \int_{\Gamma} \log(u(g)) d\mu(g) \right\| \leq \frac{r^2}{2(1 - r)} + \int_{\Gamma} \|u(g) - 1\| d\mu(g) \leq 2r.$$

We therefore get, integrating and using (2.2),

$$\left\| \exp \left(\int_{\Gamma} \log(u(g)) d\mu(g) \right) - \int_{\Gamma} \log(u(g)) d\mu(g) - 1 \right\| \leq \frac{(2r)^2}{2(1 - 2r)}.$$

Combining this estimate with (2.3) gives

$$\left\| \exp \left(\int_{\Gamma} \log(u(g)) d\mu(g) \right) - \int_{\Gamma} u(g) d\mu(g) \right\| \leq \frac{r^2}{2(1 - r)} + \frac{(2r)^2}{2(1 - 2r)} \leq \frac{5r^2}{2(1 - 2r)},$$

as desired. \square

Lemma 2.3. Let Γ be a compact group with normalized Haar measure μ . Let A be a unital C^* -algebra. Suppose $r \in [0, \frac{1}{5}]$, and let $\rho: \Gamma \rightarrow U(A)$ be a continuous function such that for all $g, h \in \Gamma$ we have

$$\|\rho(gh) - \rho(g)\rho(h)\| \leq r.$$

For $g \in \Gamma$ define

$$\sigma(g) = \exp \left(\int_{\Gamma} \log \left(\rho(k)^* \rho(kg) \rho(g)^* \right) d\mu(k) \right) \rho(g).$$

Then σ is a continuous function from Γ to $U(A)$ which satisfies

$$\|\sigma(gh) - \sigma(g)\sigma(h)\| \leq 17r^2 \quad \text{and} \quad \|\sigma(g) - \rho(g)\| \leq 2r$$

for all $g, h \in \Gamma$.

Proof. For $g \in \Gamma$, define

$$\sigma_0(g) = \int_{\Gamma} \rho(k)^* \rho(kg) d\mu(k).$$

The first part of the proof of Proposition 5.14 of [1] shows that for $g, h \in \Gamma$, we have

$$\|\sigma_0(gh) - \sigma_0(g)\sigma_0(h)\| \leq 2r^2 \quad \text{and} \quad \|\sigma_0(g) - \rho(g)\| \leq r.$$

The rest of the proof in [1] uses a Lie algebra valued logarithm, called “ln” there. We replace statements in [1] involving the Lie algebra of the codomain with the use of the logarithm coming from holomorphic functional calculus. Rewriting

$$\sigma_0(g) = \left(\int_{\Gamma} \rho(k)^* \rho(kg) \rho(g)^* d\mu(k) \right) \rho(g)$$

and applying Lemma 2.2, we get

$$\|\sigma(g) - \sigma_0(g)\| \leq \frac{5r^2}{2(1-2r)} \leq 5r^2 \leq r$$

for all $g \in \Gamma$. This implies $\|\sigma(g) - \rho(g)\| \leq 2r$ for all $g \in \Gamma$, which is the second of the required estimates. Clearly $\|\sigma(g)\| \leq 1$ for all $g \in \Gamma$. Lemma 2.2 implies $\|\sigma_0(g)\| \leq 1$ for all $g \in \Gamma$. Therefore

$$\begin{aligned} & \|\sigma(gh) - \sigma(g)\sigma(h)\| \\ & \leq \|\sigma(gh) - \sigma_0(gh)\| + \|\sigma(g) - \sigma_0(g)\| + \|\sigma(h) - \sigma_0(h)\| + \|\sigma_0(gh) - \sigma_0(g)\sigma_0(h)\| \\ & \leq 5r^2 + 5r^2 + 5r^2 + 2r^2 = 17r^2. \end{aligned}$$

This is the first of the required estimates. \square

Lemma 2.4. Let Γ be a compact group with normalized Haar measure μ . Let A and B be unital C^* -algebras, and let $\kappa: A \rightarrow B$ be a unital homomorphism. Suppose $0 \leq r < \frac{1}{17}$, and let $\rho_0: \Gamma \rightarrow U(A)$ be a continuous map such that for all $g, h \in \Gamma$, we have

$$\|\rho_0(gh) - \rho_0(g)\rho_0(h)\| \leq r \quad \text{and} \quad (\kappa \circ \rho_0)(gh) = (\kappa \circ \rho_0)(g)(\kappa \circ \rho_0)(h).$$

Inductively define functions $\rho_m: \Gamma \rightarrow A$ by (following Lemma 2.3)

$$\rho_{m+1}(g) = \exp \left(\int_{\Gamma} \log \left(\rho_m(k)^* \rho_m(kg) \rho_m(g)^* \right) d\mu(k) \right) \rho_m(g)$$

for $g \in \Gamma$. Then for every $m \in \mathbb{Z}_{>0}$ the function ρ_m is a well defined continuous function from Γ to $U(A)$ such that $\kappa \circ \rho_m = \kappa \circ \rho_0$. Moreover, the functions ρ_m converge uniformly to a continuous homomorphism $\rho: \Gamma \rightarrow U(A)$ such that

$$\sup_{g \in \Gamma} \|\rho(g) - \rho_0(g)\| \leq \frac{2r}{1-17r} \quad \text{and} \quad \kappa \circ \rho = \kappa \circ \rho_0.$$

Proof. We claim that for all $m \in \mathbb{Z}_{\geq 0}$, the function ρ_m is well defined, continuous, take values in $U(A)$, and satisfies $\kappa \circ \rho_m = \kappa \circ \rho_0$, and that for $g, h \in \Gamma$ we have

$$(2.4) \quad \|\rho_m(gh) - \rho_m(g)\rho_m(h)\| \leq r(17r)^m$$

and

$$(2.5) \quad \|\rho_m(g) - \rho_{m-1}(g)\| \leq 2r(17r)^{m-1}.$$

The proof of the claim is by induction on m . The case $m = 1$ is Lemma 2.3 and $r \leq \frac{1}{5}$. Assume the result is known for m . Since the estimates (2.4) and (2.5) hold

for m , and by Lemma 2.3 and because $r(17r)^m < r \leq \frac{1}{5}$, the function ρ_{m+1} is well defined, continuous, take values in $U(A)$, and for $g, h \in \Gamma$ we have

$$\|\rho_{m+1}(g) - \rho_m(g)\| \leq 2r(17r)^m$$

and, also using $17r < 1$ at the last step,

$$\|\rho_{m+1}(gh) - \rho_{m+1}(g)\rho_{m+1}(h)\| \leq 17(r(17r)^m)^2 = r(17r)^{m+1}(17r)^m < r(17r)^{m+1}.$$

It remains to prove that $\kappa \circ \rho_{m+1} = \kappa \circ \rho_0$. Let $g \in \Gamma$. Using $\kappa \circ \rho_m = \kappa \circ \rho_0$ at the second step and the fact that $\kappa \circ \rho_0$ is a homomorphism at the last step, we get

$$\begin{aligned} & \kappa \left(\exp \left(\int_{\Gamma} \log \left(\rho_m(k)^* \rho_m(kg) \rho_m(g)^* \right) d\mu(k) \right) \right) \\ &= \exp \left(\int_{\Gamma} \log \left((\kappa \circ \rho_m)(k)^* (\kappa \circ \rho_m)(kg) (\kappa \circ \rho_m)(g)^* \right) d\mu(k) \right) \\ &= \exp \left(\int_{\Gamma} \log \left((\kappa \circ \rho_0)(k)^* (\kappa \circ \rho_0)(kg) (\kappa \circ \rho_0)(g)^* \right) d\mu(k) \right) = 1. \end{aligned}$$

Therefore $\kappa(\rho_{m+1}(g)) = \kappa(\rho_m(g)) = \kappa(\rho_0(g))$. This completes the induction, and proves the claim.

The estimate (2.5) implies that there is a continuous function $\rho: \Gamma \rightarrow U(A)$ such that $\rho_m \rightarrow \rho$ uniformly, and in fact for $g \in \Gamma$ we have

$$\|\rho(g) - \rho_0(g)\| \leq \sum_{m=1}^{\infty} 2r(17r)^{m-1} = \frac{2r}{1-17r}.$$

The estimate (2.4) and convergence imply that ρ is a homomorphism. Continuity of κ implies that $\kappa \circ \rho = \kappa \circ \rho_0$. \square

The following proposition is a variant of the fact that two close homomorphisms from a finite dimensional C^* -algebra are unitarily equivalent.

Proposition 2.5. Let Γ be a compact group, let A and B be unital C^* -algebras, and let $\kappa: A \rightarrow B$ be a unital homomorphism. Let $\rho, \sigma: \Gamma \rightarrow U(A)$ be two continuous homomorphisms such that

$$\|\rho(g) - \sigma(g)\| < 1 \quad \text{and} \quad \kappa \circ \rho(g) = \kappa \circ \sigma(g)$$

for all $g \in \Gamma$. Then there exists a unitary $u \in A$ such that $u\rho(g)u^* = \sigma(g)$ for all $g \in \Gamma$, and such that $\kappa(u) = 1$.

Proof. Let μ be normalized Haar measure on Γ . Define

$$a = \int_{\Gamma} \sigma(h)^* \rho(h) d\mu(h).$$

For $g \in \Gamma$ we get, changing variables at the second step,

$$(2.6) \quad a\rho(g) = \int_{\Gamma} \sigma(h)^* \rho(hg) d\mu(h) = \int_{\Gamma} \sigma(hg^{-1})^* \rho(h) d\mu(h) = \sigma(g)a.$$

Since $\|\sigma(h)^* \rho(h) - 1\| < 1$ for all $h \in \Gamma$, we have $\|a - 1\| < 1$. Therefore $u = a(a^*a)^{-1/2}$ is a well defined unitary in A . Taking adjoints in (2.6), we get $a^*\sigma(g) = \rho(g)a^*$ for all $g \in \Gamma$, so a^*a commutes with $\rho(g)$. Thus $(a^*a)^{-1/2}$ commutes with $\rho(g)$. Applying (2.6) again, we get $u\rho(g) = \sigma(g)u$ for all $g \in \Gamma$.

The hypotheses imply that $\kappa(a) = 1$, so also $\kappa(u) = 1$. \square

Theorem 2.6. Let $\alpha: G \rightarrow \text{Aut}(A)$ be an action of a compact group G on a finite dimensional C^* -algebra A . Then (G, A, α) is equivariantly semiprojective.

Proof. Set $\varepsilon_0 = \frac{1}{6 \cdot 34}$, and choose $\varepsilon > 0$ such that $\varepsilon \leq \varepsilon_0$ and such that whenever A is a unital C^* -algebra, $u \in U(A)$, and $a \in A$ satisfies $\|a - u\| < \varepsilon$, then we have $\|a(a^*a)^{-1/2} - u\| < \varepsilon_0$.

Let the notation be as in Definition 1.1 and Remark 1.3(3). Let $\varphi: A \rightarrow C/J$ be a unital equivariant homomorphism. Since finite dimensional C^* -algebras are semiprojective, there exist n_0 and a unital homomorphism (not necessarily equivariant) $\psi_0: A \rightarrow C/J_{n_0}$ which lifts φ .

For $n \geq n_0$, define $f_n: G \times U(A) \rightarrow [0, \infty)$ by

$$f_n(g, x) = \|\pi_{n, n_0}(\psi_0(\alpha_g(x)) - \gamma_g^{(n_0)}(\psi_0(x)))\|$$

for $g \in G$ and $x \in U(A)$. The functions f_n are continuous and satisfy

$$f_{n_0} \geq f_{n_0+1} \geq f_{n_0+2} \geq \cdots$$

Using $J = \overline{\bigcup_{n=1}^{\infty} J_n}$ at the first step and equivariance of φ at the second step, for $g \in G$ and $x \in U(A)$ we have

$$\lim_{n \rightarrow \infty} f_n(g, x) = \|\varphi(\alpha_g(x)) - \gamma_g(\varphi(x))\| = 0.$$

Since $G \times U(A)$ is compact, Dini's Theorem (Proposition 11 in Chapter 9 of [30]) implies that $f_n \rightarrow 0$ uniformly. Therefore there exists $n \geq n_0$ such that for all $g \in G$ and $x \in U(A)$, we have

$$\|\pi_{n, n_0}(\psi_0(\alpha_g(x)) - \gamma_g^{(n_0)}(\psi_0(x)))\| < \varepsilon.$$

Set $\psi_1 = \pi_{n, n_0} \circ \psi_0$. Then this estimate becomes

$$(2.7) \quad \|\psi_1(\alpha_g(x)) - \gamma_g^{(n)}(\psi_1(x))\| < \varepsilon.$$

for every $g \in G$ and $x \in U(A)$.

Let ν be normalized Haar measure on G . For $x \in A$ define

$$T(x) = \int_G (\gamma_h^{(n)} \circ \psi_1 \circ \alpha_h^{-1})(x) d\nu(h).$$

Then for $g \in G$ we have

$$\begin{aligned} \gamma_g^{(n)}(T(x)) &= \int_G (\gamma_{gh}^{(n)} \circ \psi_1 \circ \alpha_h^{-1})(x) d\nu(h) \\ &= \int_G (\gamma_h^{(n)} \circ \psi_1 \circ \alpha_{g^{-1}h}^{-1})(x) d\nu(h) = T(\alpha_g(x)). \end{aligned}$$

So T is equivariant. Also, since $\pi_n \circ \psi_1 = \varphi$ is equivariant, we have $\pi_n(T(x)) = \varphi(x)$ for all $x \in A$. It follows from (2.7) that $\|T(x) - \psi_1(x)\| < \varepsilon$ for all $x \in U(A)$. Since $\varepsilon < 1$, we may define $\rho_0: U(A) \rightarrow U(C/J_n)$ by

$$\rho_0(x) = T(x)(T(x)^*T(x))^{-1/2}$$

for $x \in U(A)$. Then

$$\gamma_g^{(n)}(\rho_0(x)) = \rho_0(\alpha_g(x)) \quad \text{and} \quad \pi_n(\rho_0(x)) = \varphi(x)$$

for all $g \in G$ and $x \in U(A)$. By the choice of ε , we have $\|\rho_0(x) - T(x)\| < \varepsilon_0$, whence

$$\|\rho_0(x) - \psi_1(x)\| \leq \|\rho_0(x) - T(x)\| + \|T(x) - \psi_1(x)\| < \varepsilon_0 + \varepsilon \leq 2\varepsilon_0.$$

Let $x, y \in U(A)$. Since

$$\psi_1(x), \psi_1(y) \in U(C/J_n) \quad \text{and} \quad \psi_1(xy) = \psi_1(x)\psi_1(y),$$

it follows that

$$\|\rho_0(xy) - \rho_0(x)\rho_0(y)\| < 6\varepsilon_0.$$

Let μ be normalized Haar measure on the compact group $U(A)$. Inductively define functions $\rho_m: \Gamma \rightarrow U(C/J_m)$ by (following Lemma 2.4)

$$\rho_{m+1}(x) = \rho_m(x) \exp \left(\int_{U(A)} \log \left(\rho_m(x)^* \rho_m(xy) \rho_m(y)^* \right) d\mu(y) \right)$$

for $x \in U(A)$. Since $6\varepsilon_0 = \frac{1}{34} < \frac{1}{17}$, Lemma 2.4 implies that each function ρ_m is a well defined continuous function from $U(A)$ to $U(C/J_n)$ and that $\rho(x) = \lim_{m \rightarrow \infty} \rho_m(x)$ defines a continuous homomorphism from $U(A)$ to $U(C/J_n)$ satisfying

$$\|\rho(x) - \rho_0(x)\| \leq \frac{2 \cdot 6\varepsilon_0}{1 - 17 \cdot 6\varepsilon_0} = \frac{2}{17} \quad \text{and} \quad \pi_n(\rho(x)) = \varphi(x)$$

for all $x \in U(A)$. Since homomorphisms respect functional calculus, an induction argument shows that

$$\gamma_g^{(n)}(\rho_m(x)) = \rho_m(\alpha_g(x))$$

for all $m \in \mathbb{Z}_{\geq 0}$, $g \in G$, and $x \in U(A)$. Therefore also

$$(2.8) \quad \gamma_g^{(n)}(\rho(x)) = \rho(\alpha_g(x))$$

for all $g \in G$ and $x \in U(A)$.

For $x \in U(A)$ we have

$$\|\rho(x) - \psi_1(x)\| \leq \|\rho(x) - \rho_0(x)\| + \|\rho_0(x) - \psi_1(x)\| < \frac{2}{17} + 6\varepsilon_0 < 1.$$

Since $\pi_n(\rho(x)) = \varphi(x) = \pi_n(\psi_1(x))$ for $x \in U(A)$, and since $U(A)$ is compact, Proposition 2.5 provides a unitary $w \in C/J_n$ such that $\pi_n(w) = 1$ and such that $w\psi_1(x)w^* = \rho(x)$ for all $x \in U(A)$. Define a homomorphism $\psi: A \rightarrow C/J_n$ by $\psi(a) = w\psi_1(x)w^*$ for $a \in A$. Then ψ lifts φ because $\pi_n(w) = 1$. Furthermore, ψ is equivariant by (2.8) and because $U(A)$ spans A . \square

As an immediate application, one can require that the projections in the definitions of the Rokhlin and tracial Rokhlin properties for finite groups be exactly orthogonal and exactly permuted by the group action, rather than merely being approximately permuted by the group action. We postpone the proof until after discussion equivariant stability of relations. See Proposition 5.26 and Proposition 5.27.

We can now show that tensoring with finite dimensional G -algebras preserves equivariant semiprojectivity. The proof is essentially due to Adam P. W. Sørensen, and Hannes Thiel and we are grateful to them for their permission to include it here. We begin with a lemma.

Lemma 2.7. Let A_1 and A_2 be unital C^* -algebras, and let $\varphi: A_1 \rightarrow A_2$ be a surjective homomorphism. Let F be a finite dimensional C^* -algebra, and let $\lambda_1: F \rightarrow A_1$ be a unital homomorphism. Set $\lambda_2 = \varphi \circ \lambda_1$. For $s = 1, 2$ define

$$B_s = \{a \in A_s : a \text{ commutes with } \lambda(x) \text{ for all } x \in F\}.$$

Then $\varphi|_{B_1}$ is a surjective homomorphism from B_1 to B_2 .

Proof. There are $n, r(1), r(2), \dots, r(n) \in \mathbb{Z}_{>0}$ such that $F = \bigoplus_{l=1}^n F_l$ and $F_l \cong M_{r(l)}$ for $l = 1, 2, \dots, n$. Let $(e_{j,k}^{(l)})_{j,k=1}^{r(l)}$ be a system of matrix units for F_l .

For $s = 1, 2$ define $E_s: A_s \rightarrow A_s$ by

$$(2.9) \quad E_s(a) = \sum_{l=1}^n \sum_{k=1}^{r(l)} \lambda_s(e_{k,1}^{(l)}) a \lambda_s(e_{1,k}^{(l)})$$

for $a \in A_s$. We claim that $E_s(a)$ commutes with $\lambda_s(x)$ for $a \in A_s$, $x \in F$, and $s = 1, 2$. It suffices to take $x = e_{i,j}^{(m)}$. In the product $E_s(a) \lambda_s(x)$, the terms coming from (2.9) with $l \neq m$ vanish, leaving

$$E_s(a) \lambda_s(x) = \sum_{k=1}^{r(m)} \lambda_s(e_{k,1}^{(m)}) a \lambda_s(e_{1,k}^{(m)} e_{i,j}^{(m)}) = \lambda_s(e_{i,1}^{(m)}) a \lambda_s(e_{1,j}^{(m)}).$$

Similarly, we also get $\lambda_s(x) E_s(a) = \lambda_s(e_{i,1}^{(m)}) a \lambda_s(e_{1,j}^{(m)})$, proving the claim.

The relation

$$\sum_{l=1}^n \sum_{k=1}^{r(l)} \lambda_s(e_{k,k}^{(l)}) = \lambda_s(1) = 1$$

implies that for $s = 1, 2$ and $a \in B_s$, we have $E_s(a) = a$. It is clear that $E_2 \circ \varphi = \varphi \circ E_1$.

Now let $b \in B_2$. Choose $a \in A_1$ such that $\varphi(a) = b$. Then $E_1(a) \in B_1$, and $\varphi(E_1(a)) = E_2(b) = b$. This completes the proof. \square

Theorem 2.8. Let G be a compact group, let A be a unital C^* -algebra, and let F be a finite dimensional C^* -algebra. Let $\alpha: G \rightarrow \text{Aut}(A)$ be an equivariantly semiprojective action, and let $\beta: G \rightarrow \text{Aut}(F)$ be any action. Then $\beta \otimes \alpha: G \rightarrow \text{Aut}(F \otimes A)$ is equivariantly semiprojective.

Proof. Define $\omega: F \rightarrow F \otimes A$ by $\omega(x) = x \otimes 1$. By Theorem 2.6 and Lemma 1.4, it suffices to prove that ω is equivariantly conditionally semiprojective. Let the notation be as in Definition 1.2, except with F in place of A and $F \otimes A$ in place of B . We have the diagram

$$\begin{array}{ccccc} & & & C & \\ & & & \downarrow \kappa_n & \\ & & & C/J_n & \\ & & & \downarrow \pi_n & \\ & & & C/J & \\ \lambda \nearrow & & \psi \nearrow & & \searrow \kappa \\ F & \xrightarrow{\omega} & F \otimes A & \xrightarrow{\varphi} & C/J \end{array}$$

in which the solid arrows correspond to given equivariant unital homomorphisms. We must find n and an equivariant unital homomorphism ψ which make the whole diagram commute.

Define

$$D = \{c \in C: c \text{ commutes with } \lambda(x) \text{ for all } x \in F\},$$

which is a G -invariant subalgebra of C . Define $I_n = J_n \cap D$ for $n \in \mathbb{Z}_{\geq 0}$, and set $I = J \cap D$. Then I, I_0, I_1, \dots are G -invariant ideals in D , and $I = \overline{\bigcup_{n=0}^{\infty} I_n}$.

Moreover, Lemma 2.7 implies that

$$D/I_n = \{c \in C/J_n : c \text{ commutes with } (\kappa_n \circ \lambda)(x) \text{ for all } x \in F\}$$

for $n \in \mathbb{Z}_{\geq 0}$, and that

$$(2.10) \quad D/I = \{c \in C/J : c \text{ commutes with } (\kappa \circ \lambda)(x) \text{ for all } x \in F\}.$$

Define an equivariant homomorphism $\varphi_0 : A \rightarrow C/J$ by $\varphi_0(a) = \varphi(1 \otimes a)$ for $a \in A$. By (2.10), the range of φ_0 is contained in D/I . Since A is equivariantly semiprojective, there are n and a unital equivariant homomorphism $\psi_0 : A \rightarrow D/I_n \subset C/J_n$ such that $\pi_n \circ \psi_0 = \varphi_0$.

By construction, the ranges of ψ_0 and $\kappa_n \circ \lambda$ commute, so there is a unital homomorphism $\psi_0 : A \rightarrow C/J_n$, necessarily equivariant, such that

$$\psi(x \otimes a) = (\kappa_n \circ \lambda)(x) \psi_0(a)$$

for all $x \in F$ and $a \in A$. Using $\pi_n \circ \kappa_n \circ \lambda = \varphi \circ \omega$, we get $\pi_n \circ \psi = \varphi$. \square

3. QUASIFREE ACTIONS ON CUNTZ ALGEBRAS

The purpose of this section is to prove that quasifree actions of compact groups on the Cuntz algebras \mathcal{O}_d and the extended Cuntz algebras E_d , for d finite, are equivariantly semiprojective. We begin by defining and introducing notation for quasifree actions.

Notation 3.1. Let $d \in \mathbb{Z}_{>0}$. (We allow $d = 1$, in which case $\mathcal{O}_d = C(S^1)$.) We write s_1, s_2, \dots, s_d for the standard generators of the Cuntz algebra \mathcal{O}_d . That is, we take \mathcal{O}_d to be generated by elements s_1, s_2, \dots, s_d satisfying the relations $s_j^* s_j = 1$ for $j = 1, 2, \dots, d$ and $\sum_{j=1}^d s_j s_j^* = 1$.

Notation 3.2. Let $d \in \mathbb{Z}_{>0}$. We recall the extended Cuntz algebra E_d . It is the universal unital C^* -algebra generated by d isometries with orthogonal range projections which are not required to add up to 1. (For $d = 1$, we get the Toeplitz algebra, the C^* -algebra of the unilateral shift, which here is called r_1 .) We call these isometries r_1, r_2, \dots, r_d , so that the relations are $r_j^* r_j = 1$ for $j = 1, 2, \dots, d$ and $r_j r_j^* r_k r_k^* = 0$ for $j, k = 1, 2, \dots, d$ with $j \neq k$. When d must be specified, we write $r_1^{(d)}, r_2^{(d)}, \dots, r_d^{(d)}$. We further let $\eta : E_d \rightarrow \mathcal{O}_d$ be the quotient map, defined, following Notation 3.1, by $\eta(r_j) = s_j$ for $j = 1, 2, \dots, d$.

Notation 3.3. For $\lambda = (\lambda_1, \lambda_2, \dots, \lambda_d) \in \mathbb{C}^d$, we further define $s_\lambda \in \mathcal{O}_d$ and $r_\lambda \in E_d$ by

$$s_\lambda = \sum_{j=1}^d \lambda_j s_j \quad \text{and} \quad r_\lambda = \sum_{j=1}^d \lambda_j r_j.$$

Notation 3.4. Let $d \in \mathbb{Z}_{>0}$. We let $(e_{j,k})_{j,k=1}^n$ be the standard system of matrix units in M_n . We denote by $\mu : M_d \rightarrow \mathcal{O}_d$ the injective unital homomorphism determined by $\mu(e_{j,k}) = s_j s_k^*$ for $j, k = 1, 2, \dots, d$. We further denote by $\mu_0 : M_d \oplus \mathbb{C} \rightarrow E_d$ the injective unital homomorphism determined by $\mu_0((e_{j,k}, 0)) = r_j r_k^*$ for $j, k = 1, 2, \dots, d$ and $\mu_0(0, 1) = 1 - \sum_{j=1}^d r_j r_j^*$.

Recall (Notation 2.1) that $U(A)$ is the unitary group of A .

Notation 3.5. Let A be a unital C^* -algebra, and let $u \in U(A)$. We denote by $\text{Ad}(u)$ the automorphism $\text{Ad}(u)(a) = uau^*$ for $a \in A$. Further let G be a topological group, and let $\rho: G \rightarrow U(A)$ be a continuous homomorphism. We denote by $\text{Ad}(\rho)$ the action $\alpha: G \rightarrow \text{Aut}(A)$ given by $\text{Ad}(\rho)_g = \text{Ad}(\rho(g))$ for $g \in G$. For $\rho: G \rightarrow U(M_d)$, we also write $\text{Ad}(\rho \oplus 1)$ for the action $g \mapsto \text{Ad}(\rho(g), 1)$ on $M_d \oplus \mathbb{C}$. We always take $M_d \oplus \mathbb{C}$ to have this action.

We now give the basic properties of quasifree actions on E_d .

Lemma 3.6. Let G be a topological group, let $d \in \mathbb{Z}_{>0}$, and let $\rho: G \rightarrow U(M_d)$ be a unitary representation of G on \mathbb{C}^d . Then there exists a unique action $\alpha^\rho: G \rightarrow \text{Aut}(E_d)$ such that, with μ_0 as in Notation 3.4, for $j = 1, 2, \dots, d$ and $g \in G$ we have

$$\alpha_g^\rho(r_j) = \mu_0(\rho(g), 1)r_j.$$

Moreover, this action has the following properties:

- (1) For all $g \in G$, if we write

$$\rho(g) = \sum_{j,k=1}^d \rho_{j,k}(g)e_{j,k} = \begin{pmatrix} \rho_{1,1}(g) & \rho_{1,2}(g) & \cdots & \rho_{1,d}(g) \\ \rho_{2,1}(g) & \rho_{2,2}(g) & \cdots & \rho_{2,d}(g) \\ \vdots & \vdots & \ddots & \vdots \\ \rho_{d,1}(g) & \rho_{d,2}(g) & \cdots & \rho_{d,d}(g) \end{pmatrix},$$

then

$$\alpha_g^\rho(r_k) = \sum_{j=1}^d \rho_{j,k}(g)r_j$$

for $k = 1, 2, \dots, d$.

- (2) Following Notation 3.3, for every $\lambda \in \mathbb{C}^d$ and $g \in G$, we have $\alpha_g^\rho(r_\lambda) = r_{\rho(g)\lambda}$.
- (3) The homomorphism μ_0 is equivariant. (Recall the action on $M_d \oplus \mathbb{C}$ from Notation 3.5.)
- (4) The projection $1 - \sum_{j=1}^d r_j r_j^* \in E_d$ is G -invariant.

Proof. For $g \in G$, we claim that the elements $\mu_0(\rho(g), 1)r_j$ satisfy the relations defining E_d . Because $\mu_0(\rho(g), 1)$ is unitary, we have

$$(\mu_0(\rho(g), 1)r_j)^* (\mu_0(\rho(g), 1)r_k) = r_j^* r_k$$

for $j, k = 1, 2, \dots, d$. Using the definition of μ_0 at the second step, we also get

$$\begin{aligned} (3.1) \quad & \sum_{j=1}^d [\mu_0(\rho(g), 1)r_j] [\mu_0(\rho(g), 1)r_j]^* \\ &= \mu_0(\rho(g), 1) \left(\sum_{j=1}^d r_j r_j^* \right) \mu_0(\rho(g), 1) = \sum_{j=1}^d r_j r_j^*. \end{aligned}$$

It is now easy to prove the claim.

It follows that there is a unique homomorphism $\alpha_g^\rho: E_d \rightarrow E_d$ such that $\alpha_g^\rho(r_j) = \mu_0(\rho(g), 1)r_j$ for $j = 1, 2, \dots, d$. Part (4) follows from (3.1). Part (1) is just a calculation, and implies part (2) when $\lambda \in \mathbb{C}^d$ is a standard basis vector. The general case of part (2) follows by linearity.

Part (2) implies that $\alpha_{gh}^\rho(s_\lambda) = \alpha_g^\rho \circ \alpha_h^\rho(s_\lambda)$ for $g, h \in G$ and $\lambda \in \mathbb{C}$, and also $\alpha_1^\rho = \text{id}_{E_d}$, so $g \mapsto \alpha_g^\rho$ is a homomorphism to $\text{Aut}(E_d)$. Continuity of $g \mapsto \alpha_g^\rho(a)$ for $a \in E_d$ follows from the fact that it holds whenever $a = r_j$ for some j .

It remains to prove (3). For $j, k = 1, 2, \dots, d$, we have

$$\begin{aligned} \alpha_g^\rho(\mu_0(e_{j,k}, 0)) &= \alpha_g^\rho(r_j) \alpha_g^\rho(r_k^*) = \mu_0(\rho(g), 1) r_j r_k^* \mu_0(\rho(g), 1)^* \\ &= \mu_0((\rho(g), 1)(e_{j,k}, 0)(\rho(g), 1)^*) = \mu_0(\text{Ad}(\rho \oplus 1)_g(e_{j,k}, 0)), \end{aligned}$$

as desired. We must also to check the analogous equation with $(0, 1)$ in place of $(e_{j,k}, 0)$, but this is immediate from part (4). \square

Here are the corresponding properties for quasifree actions on \mathcal{O}_d . These are mostly well known, and are stated for reference and to establish notation. They also follow from Lemma 3.6.

Lemma 3.7. Let G be a topological group, let $d \in \mathbb{Z}_{>0}$, and let $\rho: G \rightarrow U(M_d)$ be a unitary representation of G on \mathbb{C}^d . Then there exists a unique action $\beta^\rho: G \rightarrow \text{Aut}(\mathcal{O}_d)$ such that, with μ as in Notation 3.4, for $j = 1, 2, \dots, d$ and $g \in G$ we have

$$\beta_g^\rho(s_j) = \mu(\rho(g), 1)s_j.$$

Moreover, this action has the following properties:

- (1) For all $g \in G$, if we write

$$\rho(g) = \sum_{j,k=1}^d \rho_{j,k}(g) e_{j,k},$$

then

$$\beta_g^\rho(s_k) = \sum_{j=1}^d \rho_{j,k}(g) s_j$$

for $k = 1, 2, \dots, d$.

- (2) Following Notation 3.3, for every $\lambda \in \mathbb{C}^d$ and $g \in G$, we have $\beta_g^\rho(s_\lambda) = s_{\rho(g)\lambda}$.
- (3) When M_d is equipped with the action $\text{Ad}(\rho)$, the homomorphism μ is equivariant.
- (4) The quotient map η from (G, E_d, α^ρ) to $(G, \mathcal{O}_d, \beta^\rho)$ is equivariant.

Proof. Lemma 3.6(4) implies that the ideal in E_d generated by $\sum_{j=1}^d r_j r_j^*$ is invariant. Therefore the quotient is a G -algebra. It is well known that we may identify $\eta: E_d \rightarrow \mathcal{O}_d$ with this quotient map. So we have an action $\beta^\rho: G \rightarrow \text{Aut}(\mathcal{O}_d)$. It is clear from the construction and the fact that $\eta(\mu_0(e_{j,k}, 0)) = \mu(e_{j,k})$ for $j, k = 1, 2, \dots, d$ that this action satisfies $\beta_g^\rho(s_j) = \mu(\rho(g), 1)s_j$ for $j = 1, 2, \dots, d$ and $g \in G$. Uniqueness of β^ρ is clear. Similarly, parts (1), (2), and (3) follow from the corresponding formulas in Lemma 3.6. \square

The algebraic computations we need for equivariant semiprojectivity of quasifree actions are contained in the following lemma.

Lemma 3.8. Let G be a topological group. Let $d \in \mathbb{Z}_{>0}$, let $\rho: G \rightarrow U(M_d)$ be a unitary representation of G on \mathbb{C}^d , and let $\alpha^\rho: G \rightarrow \text{Aut}(E_d)$ be the quasifree action of Lemma 3.6. Let $\mu_0: M_d \oplus \mathbb{C} \rightarrow E_d$ be as in Notation 3.4, and recall (Notation 3.5) the action $\text{Ad}(\rho \oplus 1)$ on $M_d \oplus \mathbb{C}$. Let (G, C, γ) be a unital G -algebra,

and let $\varphi: E_d \rightarrow C$ be a unital homomorphism such that $\varphi \circ \mu_0$ is equivariant. For $g \in G$, define

$$w(g) = \varphi(\alpha_g^\rho(r_1))^* \gamma_g(\varphi(r_1)).$$

Then:

- (1) $g \mapsto w(g)$ is a continuous function from G to $U(C)$.
- (2) For $j = 1, 2, \dots, d$ and $g \in G$, we have $\varphi(\alpha_g^\rho(r_j))w(g) = \gamma_g(\varphi(r_j))$.
- (3) For every $g, h \in G$, we have $w(gh) = w(g)\gamma_g(w(h))$.
- (4) For every $g \in G$, we have $\|w(g) - 1\| = \|\varphi(\alpha_g^\rho(r_1)) - \gamma_g(\varphi(r_1))\|$.
- (5) If $v \in U(C)$ satisfies $v\gamma_g(v)^* = w(g)$ for all $g \in G$, then there is a unique unital homomorphism $\psi: E_d \rightarrow C$ such that $\psi(r_j) = \varphi(r_j)v$ for $j = 1, 2, \dots, d$. Moreover, ψ is equivariant and $\psi \circ \mu_0 = \varphi \circ \mu_0$.
- (6) If $\kappa: C \rightarrow D$ is an equivariant homomorphism from C to some other G -algebra D , and $\kappa \circ \varphi$ is equivariant, then $\kappa(w(g)) = 1$ for all $g \in G$.

Proof. We use the usual notation for matrix units, as in Notation 3.4. We also recall (Lemma 3.6(3)) that μ_0 is equivariant.

We prove (1) by showing that $\varphi(\alpha_g^\rho(r_1))$ and $\gamma_g(\varphi(r_1))$ are isometries with the same range projection. It is clear that both are isometries. The range projections are

$$\varphi(\alpha_g^\rho(r_1))\varphi(\alpha_g^\rho(r_1))^* = \varphi(\alpha_g^\rho(r_1 r_1^*)) = (\varphi \circ \mu_0)(\text{Ad}(\rho \oplus 1)_g(e_{1,1}, 0))$$

and

$$\gamma_g(\varphi(r_1))\gamma_g(\varphi(r_1))^* = \gamma_g(\varphi(r_1 r_1^*)) = \gamma_g((\varphi \circ \mu_0)(e_{1,1}, 0)).$$

These are equal because $\varphi \circ \mu_0$ is equivariant. So (1) follows.

For (2), we have, using equivariance of both μ_0 and $\varphi \circ \mu_0$ at the third step,

$$\begin{aligned} \varphi(\alpha_g^\rho(r_j))w_g &= \varphi(\alpha_g^\rho(r_j))\varphi(\alpha_g^\rho(r_1))^* \gamma_g(\varphi(r_1)) = (\varphi \circ \alpha_g^\rho \circ \mu_0)(e_{j,1}, 0)(\gamma_g \circ \varphi)(r_1) \\ &= (\gamma_g \circ \varphi \circ \mu_0)(e_{j,1}, 0)(\gamma_g \circ \varphi)(r_1) = (\gamma_g \circ \varphi)(r_j r_1^* r_1) = (\gamma_g \circ \varphi)(r_j), \end{aligned}$$

as desired.

For (3), we simplify the notation by defining

$$(3.2) \quad u(g) = (\varphi \circ \mu_0)(\rho(g), 1)$$

for $g \in G$. Then $u(g)$ is unitary. By the definition of α^ρ , we have

$$(3.3) \quad \varphi(\alpha_g^\rho(r_j)) = u(g)\varphi(r_j)$$

for $g \in G$ and $j = 1, 2, \dots, d$. By equivariance of $\varphi \circ \mu_0$, we have

$$(3.4) \quad (\gamma_g \circ \varphi \circ \mu_0)(x) = (\text{Ad}(u(g)) \circ \varphi \circ \mu_0)(x)$$

for $g \in G$ and $x \in M_d \oplus \mathbb{C}$. Using (3.3) at the first step, (3.2) and (3.4) at the second step, $\varphi(r_1 r_1^*) = \mu_0(e_{1,1}, 0)$ and (3.4) at the third step, and $r_1 r_1^* r_1 = r_1$, (3.3), and $u(g)u(h) = u(gh)$ at the last step, for $g, h \in G$ we get

$$\begin{aligned} w(g)\gamma_g(w(h)) &= [\varphi(r_1)^* u(g)^* \gamma_g(\varphi(r_1))] \gamma_g(\varphi(r_1)^* u(h)^* \gamma_h(\varphi(r_1))) \\ &= \varphi(r_1)^* u(g)^* \gamma_g(\varphi(r_1 r_1^*)) [u(g)u(h)^* u(g)^*] \gamma_{gh}(\varphi(r_1)) \\ &= \varphi(r_1)^* \varphi(r_1 r_1^*) u(h)^* u(g)^* \gamma_{gh}(\varphi(r_1)) = w(gh). \end{aligned}$$

This proves (3).

For (4), use the fact that $\varphi(\alpha_g^\rho(r_1))$ is an isometry at the first step and part (2) at the second step to write

$$\|w(g) - 1\| = \|\varphi(\alpha_g^\rho(r_1))w(g) - \varphi(\alpha_g^\rho(r_1))\| = \|\gamma_g(\varphi(r_1)) - \varphi(\alpha_g^\rho(r_1))\|.$$

We prove (5). Existence and uniqueness of ψ are true for any unitary v , because the elements $\varphi(r_j)v$ are isometries with orthogonal ranges. For $j, k = 1, 2, \dots, d$, we have

$$(\psi \circ \mu_0)(e_{j,k}, 0) = \psi(r_j)\psi(r_k^*) = (\varphi(r_j)v)(v^*\psi(r_k^*)) = (\varphi \circ \mu_0)(e_{j,k}, 0).$$

Since also $(\psi \circ \mu_0)(1) = (\varphi \circ \mu_0)(1)$, it follows that $\psi \circ \mu_0 = \varphi \circ \mu_0$.

It remains to prove that ψ is equivariant. In the following calculation, we let $u(g)$ be as in (3.2). We use (2) and (3.3) at step 3, and (3.2) and $\psi \circ \mu_0 = \varphi \circ \mu_0$ at step 5, to get, for $g \in G$ and $j = 1, 2, \dots, d$,

$$\begin{aligned} \gamma_g(\psi(r_j)) &= \gamma_g(\varphi(r_j))\gamma_g(v) = \gamma_g(\varphi(r_j))w(g)^*v \\ &= u(g)\varphi(r_j)v = u(g)\psi(r_j) = \psi(\mu_0(\rho(g), 1)r_j) = \psi(\alpha_g^\rho(r_j)). \end{aligned}$$

Equivariance of ψ follows.

Part (6) is immediate. \square

Lemma 3.8 will be used to produce cocycles which are close to 1. To deal with them, we need results similar to Lemma 2.3 and Lemma 2.4.

Lemma 3.9. Let G be a compact group with normalized Haar measure μ . Let (G, A, α) be a unital G -algebra, and let $w: G \rightarrow U(A)$ be a continuous function such that for all $g, h \in G$ we have $w(gh) = w(g)\alpha_g(w(h))$. Suppose $r \in [0, \frac{1}{5}]$, and let $v \in U(A)$ satisfy

$$\|v\alpha_g(v)^* - w(g)\| \leq r$$

for all $g \in G$. Define

$$z = v \exp \left(\int_G \log \left(v^* \alpha_h^{-1}(w(h)^*v) \right) d\mu(h) \right).$$

Then $z \in U(A)$ and satisfies

$$\|z\alpha_g(z)^* - w(g)\| \leq 10r^2$$

for all $g \in G$ and

$$\|z - v\| \leq 2r.$$

Proof. For every $h \in G$, we have

$$(3.5) \quad \|v^* \alpha_h^{-1}(w(h)^*v) - 1\| = \|[w(h) - v\alpha_h(v)^*]^*\| \leq r.$$

Since $r < 1$, the logarithm in the formula for v exists, so v is well defined. Moreover,

$$\log \left(v^* \alpha_h^{-1}(w(h)^*v) \right) \in iA_{\text{sa}}$$

for $h \in G$, so $v \in U(A)$ implies $z \in U(A)$.

Define

$$z_0 = \int_G \alpha_h^{-1}(w(h)^*v) d\mu(h).$$

Using $\|v\alpha_h(v)^* - w(h)\| \leq r$ at the third step, we get

$$\|z_0 - v\| \leq \int_G \|\alpha_h^{-1}(w(h)^*v) - v\| d\mu(h) = \int_G \|w(h) - \alpha_h(v)v^*\| d\mu(h) \leq r.$$

Then, making the change of variables h to hg at the first step and using $w(hg) = w(h)\alpha_h(w(g))$ at the second step, for $g \in G$ we have

$$\alpha_g(z_0) = \int_G \alpha_h^{-1}(w(h)^*v) d\mu(h) = \int_G w(g)^* \alpha_h^{-1}(w(h)^*v) d\mu(h) = w(g)^* z_0.$$

Rewriting

$$z_0 = v \int_G v^* \alpha_h^{-1}(w(h)^* v) d\mu(h)$$

and using (3.5), we can apply Lemma 2.2 to get

$$\|z - z_0\| \leq \frac{5r^2}{2(1-2r)} \leq 5r^2.$$

It follows from the equation $\alpha_g(z_0) = w(g)^* z_0$ that $\|\alpha_g(z) - w(g)^* z\| \leq 10r^2$. The first estimate in the conclusion follows.

Since $r \leq \frac{1}{5}$ and we already know $\|z_0 - v\| \leq r$, we also get $\|z - v\| \leq 2r$. This is the second estimate in the conclusion. \square

Lemma 3.10. Let G be a compact group with normalized Haar measure μ . Let (G, A, α) and (G, B, β) be unital G -algebras, and let $\kappa: A \rightarrow B$ be a unital equivariant homomorphism. Let $w: G \rightarrow U(A)$ be a continuous function such that for all $g, h \in G$ we have $w(gh) = w(g)\alpha_g(w(h))$. Suppose $0 \leq r < \frac{1}{10}$, and let $v_0 \in U(A)$ satisfy

$$\|v_0 \alpha_g(v_0)^* - w(g)\| \leq r \quad \text{and} \quad \kappa(v_0) \beta_g(\kappa(v_0))^* = \kappa(w(g))$$

for all $g \in G$. Inductively define $v_m \in U(A)$ by (following Lemma 3.9)

$$v_{m+1} = v_m \exp \left(\int_G \log \left(v_m^* \alpha_h^{-1}(w(h)^* v_m) \right) d\mu(h) \right).$$

Then for every $m \in \mathbb{Z}_{>0}$ the element v_m is a well defined unitary in A such that $\kappa(v_m) = \kappa(v)$. Moreover, $v = \lim_{m \rightarrow \infty} v_m$ exists and satisfies $v \alpha_g(v)^* = w(g)$ for all $g \in G$, and also

$$\|v - v_0\| \leq \frac{2r}{1-10r} \quad \text{and} \quad \kappa(v) = \kappa(v_0).$$

Proof. The proof is essentially the same as the proof of Lemma 2.4. One proves by induction that for all $m \in \mathbb{Z}_{\geq 0}$, the element v_m is well defined, in $U(A)$, and satisfies

$$\kappa(v_m) = \kappa(v) \quad \text{and} \quad \|v_m - v_{m-1}\| \leq 2r(10r)^{m-1},$$

and that for $g \in G$ we have

$$\|v_m \alpha_g(v_m)^* - w(g)\| \leq r(10r)^m.$$

We omit further details. \square

Theorem 3.11. Let G be a compact group, let $d \in \mathbb{Z}_{>0}$, and let $\rho: G \rightarrow U(M_d)$ be a unitary representation of G on \mathbb{C}^d . Then the quasifree action $\alpha^\rho: G \rightarrow \text{Aut}(E_d)$ of Lemma 3.6 is equivariantly semiprojective.

Proof. Let $\mu_0: M_d \oplus \mathbb{C} \rightarrow E_d$ be as in Notation 3.4. Recall (Notation 3.5) the action $\text{Ad}(\rho \oplus 1)$ on $M_d \oplus \mathbb{C}$. Then μ_0 is equivariant by Lemma 3.6(3). The action $\text{Ad}(\rho \oplus 1)$ is equivariantly semiprojective by Theorem 2.6. By Lemma 1.4, it therefore suffices to prove that μ_0 is equivariantly conditionally semiprojective in the sense of Definition 1.2.

We adopt the notation of Definition 1.2 and Remark 1.3. Thus, assume that $\lambda: M_d \oplus \mathbb{C} \rightarrow C$ and $\varphi: B \rightarrow C/J$ are unital equivariant homomorphisms such that $\kappa \circ \lambda = \varphi \circ \omega$. Since E_d is semiprojective without the group, there exists $n_0 \in \mathbb{Z}_{>0}$ and a unital homomorphism $\nu_0: E_d \rightarrow C/J_{n_0}$ such that $\pi_{n_0} \circ \nu_0 = \varphi$. In particular,

$$\pi_{n_0} \circ \nu_0 \circ \mu_0 = \pi_{n_0} \circ \kappa_{n_0} \circ \lambda.$$

For $k \geq n_0$ define $f_k: U(M_d \oplus \mathbb{C}) \rightarrow [0, \infty)$ by

$$f_k(x) = \|\pi_{k,n_0}((\nu_0 \circ \mu_0)(x) - (\kappa_{n_0} \circ \lambda)(x))\|.$$

for $x \in U(M_d \oplus \mathbb{C})$. The functions f_n are continuous, and satisfy

$$f_{n_0} \geq f_{n_0+1} \geq f_{n_0+2} \geq \cdots$$

and $f_k \rightarrow 0$ pointwise. Since $U(M_d \oplus \mathbb{C})$ is compact, Dini's Theorem (Proposition 11 in Chapter 9 of [30]) implies that $f_k \rightarrow 0$ uniformly. Therefore there exists $n_1 \geq n_0$ such that for all $x \in U(A)$, we have

$$\|\pi_{n_1,n_0}((\nu_0 \circ \mu_0)(x) - (\kappa_{n_0} \circ \lambda)(x))\| < \frac{1}{2}.$$

Proposition 2.5 provides a unitary $u \in U(C/J_{n_1})$ such that $\pi_{n_1}(u) = 1$ and

$$u(\pi_{n_1,n_0} \circ \nu_0 \circ \mu_0)(x)u^* = (\pi_{n_1,n_0} \circ \kappa_{n_0} \circ \lambda)(x)$$

for all $x \in U(M_d \oplus \mathbb{C})$. Define $\nu_1: E_d \rightarrow C/J_{n_1}$ by $\nu_1 = \text{Ad}(u) \circ \pi_{n_1,n_0} \circ \nu_0$. Then $\pi_{n_1} \circ \nu_1 = \varphi$ and $\kappa_{n_1} \circ \lambda = \nu_1 \circ \mu_0$.

For $k \geq n_1$, the functions

$$g \mapsto \|\pi_{k,n_1}((\gamma_g^{(k)} \circ \nu_1)(r_1) - (\nu_1 \circ \alpha_g^\rho)(r_1))\|$$

are continuous and pointwise nonincreasing as $k \rightarrow \infty$. Since π_{n_1} and $\pi_{n_1} \circ \nu_1 = \varphi$ are equivariant, these functions converge pointwise to zero. Another application of Dini's Theorem provides $n \geq n_1$ such that, with $\nu = \pi_{n,n_1} \circ \nu_1$ and using equivariance of π_{n,n_1} , we have

$$\sup_{g \in G} \|(\gamma_g^{(n)} \circ \nu)(r_1) - (\nu \circ \alpha_g^\rho)(r_1)\| < \frac{1}{20}.$$

Now let $w(g)$ be as in Lemma 3.8, with C/J_n in place of C and ν in place of φ . Then $\sup_{g \in G} \|w(g) - 1\| < \frac{1}{20}$ by Lemma 3.8(4) and $\pi_n(w(g)) = 1$ for all $g \in G$ by Lemma 3.8(6). Using these facts, Lemma 3.8(1), and the cocycle condition of Lemma 3.8(3), we can apply Lemma 3.10 with $v_0 = 1$ to find $v \in U(C/J_n)$ such that $\pi_n(v) = 1$ and $v\alpha_g(v)^* = w(g)$ for all $g \in G$. Let $\psi: E_d \rightarrow C/J_n$ be as in Lemma 3.8(5) with this choice of v . Then ψ is equivariant and $\psi \circ \mu_0 = \nu \circ \mu_0$ by Lemma 3.8(5). Since $\nu_1 \circ \mu_0 = \kappa_{n_1} \circ \lambda$, we get $\psi \circ \mu_0 = \kappa_n \circ \lambda$. Since $\pi_n(v) = 1$, we get $\pi_n(\psi(r_j)) = \pi_n(\nu(r_j)) = \varphi(r_j)$ for $j = 1, 2, \dots, d$. Therefore $\pi_n \circ \psi = \varphi$. This completes the proof that μ_0 is equivariantly conditionally semiprojective. \square

Corollary 3.12. Let G be a compact group, let $d \in \mathbb{Z}_{>0}$, and let $\rho: G \rightarrow U(M_d)$ be a unitary representation of G on \mathbb{C}^d . Then the quasifree action $\beta^\rho: G \rightarrow \text{Aut}(\mathcal{O}_d)$ of Lemma 3.7 is equivariantly semiprojective.

Proof. By Theorem 3.11 and Lemma 1.4, it suffices to prove that the quotient map $\eta: E_d \rightarrow \mathcal{O}_d$ is equivariantly conditionally semiprojective in the sense of Definition 1.2.

Let the notation be as in Definition 1.2, except that the map called ω there is η . Set $f = \lambda \left(1 - \sum_{j=1}^d r_j r_j^*\right)$, which is a projection in C . Then

$$\kappa(f) = \varphi \left(1 - \sum_{j=1}^d s_j s_j^*\right) = 0.$$

Therefore there is $n \in \mathbb{Z}_{>0}$ such that $\|\kappa_n(f)\| < 1$. Since $\kappa_n(f)$ is a projection, this means that $\kappa_n(f) = 0$, that is, $(\kappa_n \circ \lambda) \left(1 - \sum_{j=1}^d r_j r_j^*\right) = 0$. Therefore there is

$\psi: \mathcal{O}_d \rightarrow C/J_n$ such that $\kappa_n \circ \lambda = \psi \circ \eta$. Since η is surjective, equivariance of ψ follows from equivariance of η and $\kappa_n \circ \lambda$. Similarly, from $\pi_n \circ \psi \circ \eta = \kappa \circ \lambda = \varphi \circ \eta$ we get $\pi_n \circ \psi = \varphi$. \square

Remark 3.13. An important example of a quasifree action is the one coming from the regular representation of a finite group. In this case, one can prove equivariant semiprojectivity without using any of the machinery developed in this section.

Let $d = \text{card}(G)$. We discuss only E_d , but the result for \mathcal{O}_d can be treated the same way, or reduced to the result for E_d as in Corollary 3.12. Label the generators of E_d as r_g for $g \in G$, and consider the action $\alpha: G \rightarrow \text{Aut}(E_d)$ determined by $\alpha_g(r_h) = r_{gh}$ for $g, h \in G$. Following the beginning of the proof of Theorem 3.11, we reduce to the situation in which we have a nonequivariant lifting $\nu: E_d \rightarrow C/J_n$ of φ and an isometry $v_1 \in C/J_n$ such that $v_1 v_1^* = (\kappa_n \circ \mu_0)(e_{1,1}, 0)$ and $\|\pi_n(v_1) - \varphi(r_1)\|$ is small. Functional calculus arguments can be used to replace v_1 by a nearby partial isometry w_1 such that $w_1 w_1^* = (\kappa_n \circ \mu_0)(e_{1,1}, 0)$ and $\pi_n(w_1) = \varphi(r_1)$. Then there is a unique homomorphism $\psi: E_d \rightarrow C/J_n$ such that $\psi(r_g) = \gamma_g^{(n)}(r_1)$, and this homomorphism is easily seen to be an equivariant lifting of φ which satisfies $\kappa_n \circ \lambda = \psi \circ \mu_0$.

Remark 3.14. We describe what happens when $d = 1$. In this case, \mathcal{O}_d becomes $C(S^1)$ and E_d becomes the C^* -algebra $C^*(s)$ of the unilateral shift s . Quasifree actions are those that factor through the action of S^1 on $C(S^1)$ coming from the translation action of S^1 on S^1 , and those that factor through the action of S^1 on $C^*(s)$ coming from the automorphisms determined by $\beta_\zeta(s) = \zeta s$ for $\zeta \in S^1$.

Thus, for example, we conclude that the translation action of S^1 on $C(S^1)$ is equivariantly semiprojective. This, however, is easy to prove directly. A unital equivariant homomorphism from $C(S^1)$ with translation to C/J with the action $\gamma^{(\infty)}$ is just a unitary $u \in C/J$ such that $\gamma_\zeta^{(\infty)}(u) = \zeta u$ for all $\zeta \in S^1$. This unitary can be partially lifted to a unitary v in some C/J_n such that $\|\gamma_\zeta^{(n)}(v) - \zeta v\|$ is small for all $\zeta \in S^1$. To get an exactly equivariant lift w , set

$$a = \int_{S^1} \zeta^{-1} \gamma_\zeta^{(n)}(v) d\zeta$$

(using normalized Haar measure on S^1), and take $w = a(a^*a)^{-1/2}$.

4. QUASIFREE ACTIONS ON \mathcal{O}_∞

The purpose of this section is to prove that quasifree actions of finite groups on the Cuntz algebras \mathcal{O}_∞ are equivariantly semiprojective. We begin with a discussion of quasifree actions on \mathcal{O}_∞ . We will need to include a point of view different from that of Lemma 3.6 and Lemma 3.7, primarily to take advantage of the KK-theory computations in [27].

Notation 4.1. For $d \in \mathbb{Z}_{>0}$, we make \mathbb{C}^d into a Hilbert space in the standard way. For $d = \infty$, we take $\mathbb{C}^d = \ell^2(\mathbb{Z}_{>0})$. We let $\delta_j \in \mathbb{C}^d$ (for $j = 1, 2, \dots, d$ or for $d \in \mathbb{Z}_{>0}$) denote the j th standard basis vector, and we let U_d be the group of unitary operators on \mathbb{C}^d , with the strong operator topology. For convenience, we also define $E_\infty = \mathcal{O}_\infty$, and denote its generators by $r_j^{(\infty)}$ as well as by s_j .

Of course, when d is finite, the topology on U_d is the same as the norm topology. We warn that the notation \mathbb{C}^∞ conflicts with notation often used for the product

or the algebraic direct sum (we are using the Hilbert direct sum), and that U_∞ conflicts with notation sometimes used for the (much smaller) algebraic direct limit of the groups U_d .

We summarize various results from [27], and relate them to the viewpoint of Lemma 3.6 and Lemma 3.7. For a C^* -algebra A , a Hilbert A - A bimodule F is as described at the beginning of Section 1 of [27]: F is a right Hilbert A -module, with A -valued scalar product which is conjugate linear in the first variable, together with an injective homomorphism $\varphi: A \rightarrow L(F)$.

Theorem 4.2 (Pimsner). Let $d \in \mathbb{Z}_{>0} \cup \{\infty\}$, and follow Notation 4.1. Make \mathbb{C}^d into a Hilbert \mathbb{C} - \mathbb{C} bimodule F_d , as described above in the obvious way, with $\varphi(\lambda) = \lambda \cdot 1_{L(F_d)}$ for $\lambda \in \mathbb{C}$. Let \mathcal{T}_d be the associated Toeplitz algebra \mathcal{T}_{F_d} as described in Definition 1.1 of [27], and call its generators T_ξ as there. Then:

- (1) There is a unique continuous action $\gamma^{(d)}: U_d \rightarrow \text{Aut}(\mathcal{T}_d)$ which satisfies $\gamma_u^{(d)}(T_\xi) = T_{u\xi}$ for all $\xi \in \mathbb{C}^d$.
- (2) There is a unique isomorphism $\sigma_d: E_d \rightarrow \mathcal{T}_d$ such that $\sigma_d(r_j^{(d)}) = T_{\delta_j}$ for all j . (Recall that for $d = \infty$ this means $\sigma_\infty: \mathcal{O}_\infty \rightarrow \mathcal{T}_\infty$.)
- (3) If $d < \infty$, then σ_d is equivariant for the action of U_d on E_d gotten by taking $\rho = \text{id}_{U_d}$ in Lemma 3.6 and the action of part (1).
- (4) If $d = \infty$, when $u \in U_\infty$ is written as a matrix $u = (u_{j,k})_{j,k=1}^\infty$, we have

$$(\sigma_d^{-1} \circ \gamma_u^{(d)} \circ \sigma_d)(r_k^{(d)}) = \sum_{j=1}^d u_{j,k} r_j^{(d)}$$

for all $k \in \mathbb{Z}_{>0}$, with convergence in the norm topology on the right.

- (5) Let $d_1 \in \mathbb{Z}_{>0}$ and $d_2 \in \mathbb{Z}_{>0} \cup \{\infty\}$ satisfy $d_1 \leq d_2$, and set $G = U_{d_1} \times U_{d_2-d_1}$. Let G act on E_{d_1} by projection to the first factor followed by the action on E_{d_1} corresponding to $\gamma^{(d)}$, and let G act on E_{d_2} by the inclusion of G in U_{d_2} as block diagonal matrices followed by the action on E_{d_2} corresponding to $\gamma^{(d)}$. Then the standard inclusion of E_{d_1} in E_{d_2} is equivariant.

Proof. The group action of (1) is obtained as in Remark 1.2(2) of [27]. In [27], for a general Hilbert bimodule F , only the action on the quotient \mathcal{O}_F of \mathcal{T}_F is described, but the same reasoning also gives an action on \mathcal{T}_F . Continuity of the action is easily checked on the generators T_ξ for $\xi \in \mathbb{C}^d$, and continuity on the algebra follows by a standard argument.

For part (2), relations giving \mathcal{T}_d as a universal C^* -algebra are described at the beginning of Section 3 of [27]. By comparing these relations with those for E_d , one sees that the maps σ_d exist and are isomorphisms.

Part (3) is a computation. For part (4), orthogonality of the ranges of the s_j shows that if $\lambda \in \mathbb{C}^\infty$ satisfies $\lambda_j = 0$ for all but finitely many $j \in \mathbb{Z}_{>0}$, then

$$\left\| \sum_{j=1}^\infty \lambda_j s_j \right\| = \|\lambda\|_2.$$

Since for all $k \in \mathbb{Z}_{>0}$,

$$\sum_{j=1}^\infty |u_{j,k}|^2 < \infty,$$

this implies convergence on the right in the formula in (4). The validity of the formula is now a computation like that for part (3).

Part (5) now follows by comparing the formulas for the actions from parts (3) and (4) with the definitions of σ_{d_1} and σ_{d_2} . \square

Remark 4.3. Following Theorem 4.2(2), we will identify \mathcal{T}_∞ with $E_\infty = \mathcal{O}_\infty$, and for finite d we will identify \mathcal{T}_d with E_d . We then write the action $\gamma^{(d)}$ of Theorem 4.2(1) as an action on E_d .

Definition 4.4. Let G be a topological group. A *quasifree action* of G on \mathcal{O}_∞ is an action of the form $\gamma^{(\infty)} \circ \rho$ with $\gamma^{(\infty)}$ as in Remark 4.3 and for some continuous homomorphism $\rho: G \rightarrow U_\infty$ (that is, for some unitary representation ρ of G on $l^2(\mathbb{Z}_{>0})$).

Remark 4.5. Theorem 4.2(3) implies that for $d \in \mathbb{Z}_{>0}$, an action of a topological group G on E_d is quasifree in the sense of Lemma 3.6 if and only if it factors through $\gamma^{(d)}: U_d \rightarrow \text{Aut}(E_d)$ in a similar way.

Proposition 4.6. Let G be a topological group, and let $\rho_1, \rho_2: G \rightarrow U_\infty$ be unitarily equivalent representations. Then the corresponding quasifree actions of G on \mathcal{O}_∞ are conjugate.

Proof. This is immediate from parts (1) and (2) of Theorem 4.2. \square

Theorem 4.7 (Pimsner). Let $d \in \mathbb{Z}_{>0} \cup \{\infty\}$, let G be a second countable locally compact group, and let $\rho: G \rightarrow U_d$ be a unitary representation. With the action of G on E_d as in Remark 4.3, the inclusion of \mathbb{C} in E_d via $\lambda \mapsto \lambda \cdot 1$ is a KK^G -equivalence.

Proof. See Theorem 4.4 and Remark 4.10(2) in [27]. \square

Definition 4.8. Let G be a topological group, and let $\rho: G \rightarrow U_\infty$ be a unitary representation of G on $l^2(\mathbb{Z}_{>0})$. We say that ρ is *filtered* if there are $d(1) < d(2) < \dots$ in $\mathbb{Z}_{>0}$ such that for each k , the projection p_k on the span of the first $d(k)$ standard basis vectors in $l^2(\mathbb{Z}_{>0})$ is G -invariant. We call the $d(k)$ -dimensional representations ρ_k given by $g \mapsto \rho(g)|_{p_k l^2(\mathbb{Z}_{>0})}$ the *filtering representations*. (They are, of course, not uniquely determined by ρ .)

We say that a quasifree action of G on \mathcal{O}_∞ is *filtered* if the corresponding representation ρ as in Definition 4.4 is filtered.

Remark 4.9. Let G be a topological group, and let $\alpha: G \rightarrow \text{Aut}(\mathcal{O}_\infty)$ be a quasifree action of G on \mathcal{O}_∞ coming from a filtered representation $\rho: G \rightarrow U_\infty$. Let the notation for a sequence of filtering representations be as in Definition 4.8. Let $\alpha^{(n)}: G \rightarrow \text{Aut}(E_{d(n)})$ be the quasifree action of Lemma 3.6 coming from the representation $g \mapsto \rho(g)|_{p_k l^2(\mathbb{Z}_{>0})}$. Then \mathcal{O}_∞ is the equivariant direct limit $\varinjlim E_{d(n)}$. This follows from Theorem 4.2 (using all its parts).

The following special version of a filtered representation is introduced for technical convenience.

Definition 4.10. Let G be a topological group, and let $\rho: G \rightarrow U_\infty$ be a filtered unitary representation of G on $l^2(\mathbb{Z}_{>0})$. We say that a collection $(\rho_k)_{k \in \mathbb{Z}_{>0}}$ of filtering representations is *almost even* if there exist $N_0, N \in \mathbb{Z}_{>0}$ and representations $\sigma_0: G \rightarrow U_{N_0}$ and $\sigma: G \rightarrow U_N$ such that, following the notation of Definition 4.8,

- (1) $d(n) = N_0 + nN$ for all $n \in \mathbb{Z}_{>0}$.
- (2) $\rho_1 = \sigma_0 \oplus \sigma$.

- (3) $\rho_{n+1} = \rho_n \oplus \sigma$ for all $n \in \mathbb{Z}_{>0}$.

It is important that we have equality in parts (2) and (3) of Definition 4.10, not merely unitary equivalence.

Lemma 4.11. Let G be a compact group, and let $\alpha: G \rightarrow \text{Aut}(\mathcal{O}_\infty)$ be a quasifree action of G on \mathcal{O}_∞ . Then:

- (1) The action α is conjugate to a filtered quasifree action.
- (2) If G is in fact finite, then α is conjugate to the quasifree action coming from a representation with an almost even filtration.

Part (2) can fail if the group is not finite. The regular representation of a second countable infinite compact group does not have an almost even filtration.

Proof of Lemma 4.11. For both parts, we use Proposition 4.6.

Part (1) is immediate from the fact that every unitary representation of a compact group is a direct sum of finite dimensional representations.

For part (2), we need to show that every representation $\pi: G \rightarrow U_\infty$ is unitarily equivalent to a representation ρ with an almost even filtration. Let $\tau_1, \tau_2, \dots, \tau_l$ be a set of representatives of the unitary equivalence classes of irreducible representations of G . We may assume that $\tau_1, \tau_2, \dots, \tau_{l_0}$ occur in π with finite multiplicities $m_1, m_2, \dots, m_{l_0} \in \mathbb{Z}_{\geq 0}$, and that $\tau_{l_0+1}, \tau_{l_0+2}, \dots, \tau_l$ occur with infinite multiplicity. Then $l_0 < l$. Take

$$\sigma_0 = \bigoplus_{k=1}^{l_0} m_k \cdot \tau_k, \quad \sigma = \bigoplus_{k=l_0+1}^l \tau_k, \quad \text{and} \quad \rho = \sigma_0 \oplus \sigma \oplus \sigma \oplus \dots$$

This completes the proof. \square

Proposition 4.12. Let G be a topological group, and let $\rho: G \rightarrow L(l^2(\mathbb{Z}_{>0}))$ be an injective filtered unitary representation of G . Then the corresponding quasifree action $\alpha: G \rightarrow \text{Aut}(\mathcal{O}_\infty)$ is pointwise outer, that is, α_g is outer for all $g \in G \setminus \{1\}$.

Proof. Adopt the notation of Definition 4.8. Also let $\delta_1, \delta_2, \dots$ be the standard basis vectors of $l^2(\mathbb{Z}_{>0})$. Let $g \in G \setminus \{1\}$; we will show that α_g is outer. Choose k so large that $\rho(g)|_{p_k l^2(\mathbb{Z}_{>0})}$ is nontrivial. Replacing $d(1), d(2), \dots$ by $d(k), d(k+1), \dots$, we may assume that $k = 1$. Since $\rho(g)|_{p_1 l^2(\mathbb{Z}_{>0})}$ is unitary and nontrivial, and since $p_1 l^2(\mathbb{Z}_{>0})$ is finite dimensional, there exists a unitary $u \in L(l^2(\mathbb{Z}_{>0}))$, of the form $u = u_0 + (1 - p_1)$ with u_0 a unitary in $L(p_1 l^2(\mathbb{Z}_{>0}))$, such that δ_1 is an eigenvector of $u\rho(g)u^*$ with eigenvalue $\zeta \neq 1$. Let $\sigma: G \rightarrow L(l^2(\mathbb{Z}_{>0}))$ be the representation $\sigma(g) = u\rho(g)u^*$, and let $\beta: G \rightarrow \text{Aut}(\mathcal{O}_\infty)$ be the corresponding quasifree action. It follows from Proposition 4.6 that β is conjugate to α . Therefore it suffices to show that β_g is outer. Note that $\beta_g(s_1) = \zeta s_1$.

We follow the proof of Theorem 4 of [9]. Suppose β_g is inner, and let $v \in \mathcal{O}_\infty$ be a unitary such that $\beta_g = \text{Ad}(v)$. Define $f: \mathbb{Z}_{>0} \rightarrow \mathbb{Z}_{>0} \times \mathbb{Z}_{>0}$ by $f(j, l) = 2^{j-1}(2l-1)$ for $j, l \in \mathbb{Z}_{>0}$. Define isometries $t_j \in L(l^2(\mathbb{Z}_{>0}))$ by $t_j \delta_l = \delta_{f(j,l)}$ for $j, l \in \mathbb{Z}_{>0}$. Since f is injective, there is a unital representation $\pi: \mathcal{O}_\infty \rightarrow L(l^2(\mathbb{Z}_{>0}))$ such that $\pi(s_j) = t_j$ for all $j \in \mathbb{Z}_{>0}$. Since $\pi(v)\delta_1 \in l^2(\mathbb{Z}_{>0})$ and has norm 1, we can write $\pi(v)\delta_1 = \sum_{k=1}^\infty \lambda_k \delta_k$ with $\sum_{k=1}^\infty |\lambda_k|^2 = 1$. Computations similar to those in the proof of Theorem 4 of [9] show that

$$\sum_{k=1}^\infty \lambda_k \delta_k = \pi(\beta_g(s_1))\pi(v)\delta_1 = \sum_{k=1}^\infty \zeta \lambda_k \delta_{2k-1}.$$

Compare coefficients. For $k = 1$, we get $\lambda_1 = 0$ since $\zeta \neq 1$. For $k > 1$, we get

$$\lambda_k = \zeta^{-1} \lambda_{2k-1} = \zeta^{-2} \lambda_{2(2k-1)-1} = \cdots.$$

Since $|\zeta| = 1$ and $\sum_{l=1}^{\infty} |\lambda_l|^2 < \infty$, this implies $\lambda_k = 0$. But then $\pi(v)\delta_1 = 0$, a contradiction. \square

Lemma 4.13. Let G be a topological group, let $\rho: G \rightarrow L(l^2(\mathbb{Z}_{>0}))$ be an injective filtered unitary representation of G , and let $\alpha: G \rightarrow \text{Aut}(\mathcal{O}_{\infty})$ be the corresponding quasifree action. Then $(\mathcal{O}_{\infty})^G$ is purely infinite and simple, and $K_1((\mathcal{O}_{\infty})^G) = 0$.

Proof. Since α is pointwise outer (Proposition 4.12), it follows from Theorem 3.1 of [19] that $C^*(G, \mathcal{O}_{\infty}, \alpha)$ is simple, and from Corollary 4.6 of [16] that $C^*(G, \mathcal{O}_{\infty}, \alpha)$ is purely infinite. The Proposition in [29] and its proof imply that $(\mathcal{O}_{\infty})^G$ is isomorphic to a corner in $C^*(G, \mathcal{O}_{\infty}, \alpha)$, necessarily full. Therefore $(\mathcal{O}_{\infty})^G$ is purely infinite and simple.

Theorem 4.7 implies that $K_1^G(\mathcal{O}_{\infty}) = 0$. From [17] or Theorem 2.8.3(7) of [22], we get $K_1(C^*(G, \mathcal{O}_{\infty}, \alpha)) = 0$. Since $(\mathcal{O}_{\infty})^G$ is a full corner in $C^*(G, \mathcal{O}_{\infty}, \alpha)$, it follows from Proposition 1.2 of [21] that $K_1((\mathcal{O}_{\infty})^G) = 0$. \square

Lemma 4.14. Let G be a finite group. Let $\rho: G \rightarrow U_{\infty}$ be an injective representation with an almost even filtration, for which we use the notation of Definition 4.10, and let $\alpha^{(n)}: G \rightarrow \text{Aut}(E_{d(n)})$ be as in Remark 4.9. For $m \in \mathbb{Z}_{>0}$, set

$$e_m = \sum_{j=N_0+(m-1)N+1}^{N_0+mN} s_j s_j^*.$$

Then there exists $M \in \mathbb{Z}_{>0}$ such that for all $n \geq M$, there are two isometries in $e_n(E_{d(n)})^G e_n$ with orthogonal ranges.

Proof. Let $\alpha: G \rightarrow \text{Aut}(\mathcal{O}_{\infty})$ be the corresponding quasifree action of G on \mathcal{O}_{∞} . Following Remark 4.9, we regard $E_{d(n)}$ as a subalgebra of \mathcal{O}_{∞} . Lemma 4.13 implies that $e_2(\mathcal{O}_{\infty})^G e_2$ is purely infinite and simple. It follows from Lemma 1.7 that

$$e_1(\mathcal{O}_{\infty})^G e_1 = \overline{\bigcup_{n=0}^{\infty} e_1(E_{d(n)})^G e_1}.$$

Therefore there is $M \in \mathbb{Z}_{>0}$ such that there are isometries $t_1, t_2 \in e_1(E_{d(M)})^G e_1$ with orthogonal ranges.

Now let $n \geq M$. Recall from Definition 4.10 that ρ_n is the direct sum of σ_0 and n copies of σ . Let $u \in U_{N_0+mN}$ be the permutation unitary which exchanges the first and last copies of σ . Then u commutes with $\rho_n(g)$ for all $g \in G$. Applying Lemma 3.6 to the group $\mathbb{Z} \times G$, we see that u induces a quasifree automorphism ψ of $E_{d(n)}$ which commutes with the action $\alpha^{(n)}$. Moreover, $\psi(e_1) = e_n$. Since $E_{d(M)} \subset E_{d(n)}$, the elements $\psi(t_1)$ and $\psi(t_2)$ are defined and are G -invariant isometries in $e_n(E_{d(n)})^G e_n$ with orthogonal ranges. \square

The following result is the equivariant analog of (a special case of) Lemma 3.3 of [5]. Our statement is more abstract; the concrete version, analogous to that given in [5], is rather long.

Lemma 4.15. Let G be a finite group, let $\rho: G \rightarrow U_{\infty}$ be an injective representation with an almost even filtration, and let $\alpha: G \rightarrow \text{Aut}(\mathcal{O}_{\infty})$ be the corresponding

quasifree action of G on \mathcal{O}_∞ . Let the notation be as in Definition 4.10 and Remark 4.9. In particular, $\mathcal{O}_\infty = \varinjlim_n E_{d(n)}$; call the maps of the system

$$\iota_{n,m}: E_{d(m)} \rightarrow E_{d(n)} \quad \text{and} \quad \iota_{\infty,m}: E_{d(m)} \rightarrow \mathcal{O}_\infty.$$

Let (G, A, α) be a unital G -algebra, and let $\pi: A \rightarrow \mathcal{O}_\infty$ be a surjective equivariant homomorphism. Then there exists $M \in \mathbb{Z}_{>0}$ such that for all $n \geq M$, the following holds. Let $\varphi: E_{d(n)} \rightarrow A$ be a unital equivariant homomorphism such that $\pi \circ \varphi = \iota_{\infty,n}$. Then there exists a unital equivariant homomorphism $\psi: E_{d(n+1)} \rightarrow A$ such that

$$\pi \circ \psi = \iota_{\infty,n+1} \quad \text{and} \quad \psi \circ \iota_{n+1,n-1} = \varphi \circ \iota_{n,n-1}.$$

Here is the diagram:

$$\begin{array}{ccccc}
 & & E_{d(n+1)} & & \\
 & \nearrow \iota_{n+1,n-1} & & \searrow \iota_{\infty,n+1} & \\
 E_{d(n-1)} & & & & A \xrightarrow{\pi} \mathcal{O}_\infty \\
 & \searrow \iota_{n,n-1} & & \nearrow \iota_{\infty,n} & \\
 & & E_{d(n)} & &
 \end{array}$$

ψ (dashed arrow from $E_{d(n+1)}$ to A)
 φ (solid arrow from $E_{d(n)}$ to A)

The solid arrows are given, and ψ is supposed to exist which makes the diagram commute.

Proof of Lemma 4.15. We use the names $r_j^{(d)}$ in Notation 3.2 for the generators of E_d , and we denote the standard generators of \mathcal{O}_∞ by s_1, s_2, \dots . Also recall that $d(m) = N_0 + mN$ for $m \in \mathbb{Z}_{>0}$.

Choose M as in Lemma 4.14.

Define $e_0 = \sum_{j=1}^{N_0} s_j s_j^*$, which is the projection in \mathcal{O}_∞ associated with the representation $\sigma_0: G \rightarrow U_{N_0}$. For $m \in \mathbb{Z}_{>0}$, set

$$e_m = \sum_{j=N_0+(m-1)N+1}^{N_0+mN} s_j s_j^* \quad \text{and} \quad q_m = \sum_{j=1}^{N_0+mN} s_j s_j^*.$$

Thus, e_m is the projection in \mathcal{O}_∞ associated with the m th copy of σ in the direct sum decomposition

$$\rho = \sigma_0 \oplus \sigma \oplus \sigma \oplus \dots,$$

and $q_m = \sum_{k=0}^m e_k$ is similarly associated with ρ_m .

For $k, l \in \mathbb{Z}_{>0}$, define

$$c_{k,l} = \sum_{j=1}^N s_{d(k-1)+j} (s_{d(l-1)+j})^*.$$

One easily checks that

$$c_{k,l} c_{l,k} = e_k \quad \text{and} \quad c_{k,l} = c_{l,k}^*$$

for $k, l \in \mathbb{Z}_{>0}$. We claim that $c_{k,l}$ is G -invariant. To prove the claim, for $m \in \mathbb{Z}_{>0}$ let $(e_{j,k}^{(m)})_{j,k=1}^{d(m)}$ be the standard system of matrix units in $M_{d(m)}$, and let $\mu_m: M_{d(m)} \oplus \mathbb{C} \rightarrow E_{d(m)}$ be the homomorphism called μ_0 in Notation 3.4. Recall that $E_{d(m)}$

has the action $\alpha^{(m)} = \alpha^{\rho_m}$, and equip $M_{d(m)} \oplus \mathbb{C}$ with the action $\text{Ad}(\rho_m \oplus 1)$ (Notation 3.5). Then μ_m is equivariant by Lemma 3.6(3). Now let $g \in G$. Set

$$m = \max(k, l) \quad \text{and} \quad w = \sum_{j=1}^N e_{d(k-1)+j, d(l-1)+j} \in M_{d(m)} \oplus \mathbb{C}.$$

Then w is G -invariant since it is a partial isometry which intertwines the k th and l th copies of σ in the direct sum decomposition of ρ_m . Therefore $c_{k,l} = (\iota_{\infty,m} \circ \mu_m)(w)$ is also G -invariant. The claim is proved.

Let $n \in \mathbb{Z}_{>0}$ satisfy $n \geq M$. By the choice of M using Lemma 4.14, there exist isometries $t_1, t_2 \in e_n(E_{d(n)})^G e_n$ with orthogonal ranges. Define partial isometries in $(\mathcal{O}_\infty)^G$ by

$$v_1 = \iota_{\infty,n}(t_1)^* \quad \text{and} \quad v_2 = c_{n+1,n} \iota_{\infty,n}(t_2)^*.$$

(For G -invariance of v_2 , use the claim above.) We now follow the proof of Lemma 3.3 of [5]. One checks that

$$v_1 v_1^* = e_n, \quad v_1^* v_1 = t_1 t_1^*, \quad v_2 v_2^* = e_{n+1}, \quad \text{and} \quad v_2^* v_2 = t_2 t_2^*.$$

Thus,

$$q_{n-1}, v_1^* v_1, v_2^* v_2 \quad \text{and} \quad q_{n-1}, v_1 v_1^*, v_2 v_2^*$$

are two sets of mutually orthogonal projections in $(\mathcal{O}_\infty)^G$, and the projections

$$1 - q_{n-1} - v_1^* v_1 - v_2^* v_2 \quad \text{and} \quad 1 - q_{n-1} - v_1 v_1^* - v_2 v_2^*$$

are both nonzero and have the same class in $K_0((\mathcal{O}_\infty)^G)$. Therefore, by Lemma 4.13, we can find $v_3 \in (\mathcal{O}_\infty)^G$ such that

$$v_3^* v_3 = 1 - q_{n-1} - v_1^* v_1 - v_2^* v_2 \quad \text{and} \quad v_3 v_3^* = 1 - q_{n-1} - v_1 v_1^* - v_2 v_2^*.$$

Set

$$w = v_1 + v_2 + v_3 \quad \text{and} \quad v = q + v_1 + v_2 + v_3.$$

Then w is a unitary in $(1 - q_{n-1})(\mathcal{O}_\infty)^G(1 - q_{n-1})$. Define

$$p = \sum_{j=1}^{d(n-1)} \varphi(r_j^{(d(n))}) \varphi(r_j^{(d(n))})^*,$$

which is a G -invariant projection in A such that $\pi(p) = q_{n-1}$. Proposition 1.2 of [21] and Lemma 4.13 imply that

$$K_1((1 - q_{n-1})(\mathcal{O}_\infty)^G(1 - q_{n-1})) = 0.$$

Theorem 1.9 of [7] now implies that $U((1 - q_{n-1})(\mathcal{O}_\infty)^G(1 - q_{n-1}))$ is connected. The map $A^G \rightarrow (\mathcal{O}_\infty)^G$ is surjective by Lemma 1.6. So there exists a unitary $y \in (1 - p)A^G(1 - p)$ such that $\pi(y) = w$. Set $u = p + y$, which is a unitary in A^G such that $\pi(u) = v$.

We have $u \varphi(r_j^{(d(n))}) = \varphi(r_j^{(d(n))})$ for $j = 1, 2, \dots, d(n-1)$. It is then easy to check that there is a unital homomorphism $\psi: E_{d(n+1)} \rightarrow A$ satisfying

$$\psi(r_j^{(d(n+1))}) = \begin{cases} \varphi(r_j^{(d(n))}) & j \leq d(n-1) \\ u \varphi(t_1) \varphi(r_j^{(d(n))}) & d(n-1) + 1 \leq j \leq d(n) \\ u \varphi(t_2) \varphi(r_{j-N}^{(d(n))}) & d(n) + 1 \leq j \leq d(n+1). \end{cases}$$

Clearly $\psi \circ \iota_{n+1, n-1} = \varphi \circ \iota_{n+1, n-1}$.

For $j = 1, 2, \dots, d(n)$, it is easily checked that

$$(\pi \circ \psi)(r_j^{(d(n+1))}) = \iota_{\infty, n+1}(r_j^{(d(n+1))}).$$

For $j = d(n) + 1, d(n) + 2, \dots, d(n+1)$, we have

$$\pi(u\varphi(t_2)\varphi(r_{j-N}^{(d(n))})) = v\iota_{\infty, n}(t_2)s_{j-N} = c_{n+1, n}\iota_{\infty, n}(t_2^*t_2)s_{j-N} = c_{n+1, n}s_{j-N} = s_j.$$

It follows that $\pi \circ \psi = \iota_{\infty, n+1}$.

To finish the proof, we must check that ψ is equivariant. It is enough to check equivariance on the generators. Since u is G -invariant and φ is equivariant, it is enough to check equivariance of the homomorphism $\psi_0: E_{d(n+1)} \rightarrow E_{d(n)}$ determined by

$$\psi_0(r_j^{(d(n+1))}) = \begin{cases} r_j^{(d(n))} & j \leq d(n-1) \\ t_1 r_j^{(d(n))} & d(n-1) + 1 \leq j \leq d(n) \\ t_2 r_{j-N}^{(d(n))} & d(n) + 1 \leq j \leq d(n+1). \end{cases}$$

Define $b, f \in E_{d(n+1)}$ by

$$b = \sum_{j=1}^N r_{d(n)+j}^{(d(n+1))} (r_{d(n-1)+j}^{(d(n+1))})^* \quad \text{and} \quad f = \sum_{j=1}^{d(n-1)} r_j^{(d(n+1))} (r_j^{(d(n+1))})^*.$$

Then $\iota_{\infty, n+1}(b) = c_{n+1, n}$ and $\iota_{\infty, n+1}$ is injective and equivariant, so b is G -invariant. Similarly, $\iota_{\infty, n+1}(f) = q_{n-1}$, so f is G -invariant. Also, $br_{j-N}^{(d(n+1))} = r_j^{(d(n+1))}$ for $j = d(n) + 1, d(n) + 2, \dots, d(n+1)$. Thus

$$(\iota_{n+1, n} \circ \psi_0)(r_j^{(d(n+1))}) = (f + t_1 + t_2 b)r_j^{(d(n+1))}$$

for $j = 1, 2, \dots, d(n+1)$. Since $f + t_1 + t_2 b$ is G -invariant, and since $\iota_{n+1, n}$ is injective and equivariant, it follows that ψ_0 is equivariant, as desired. This completes the proof. \square

Theorem 4.16. Let G be a finite group. Let $\alpha: G \rightarrow \text{Aut}(\mathcal{O}_\infty)$ be a quasifree action. Then α is equivariantly semiprojective.

Proof. We follow the proof of Theorem 3.2 of [5]. Let $\rho: G \rightarrow U_\infty$ be the representation which gives rise to α . Using Lemma 1.8, we may reduce to the case in which ρ is injective. By Lemma 4.11(2), we may assume that ρ has an almost even filtration as in Definition 4.10. Let the notation be as in Lemma 4.15, and choose M as there.

We follow the notation in Remark 1.3(3): C is a unital G -algebra with an increasing sequence of invariant ideals J_n , and $J = \overline{\bigcup_{n=1}^\infty J_n}$. The map $\pi_n: C/J_n \rightarrow C/J$ is the quotient map.

Let $\varphi: \mathcal{O}_\infty \rightarrow C/J$ be a unital equivariant homomorphism. First suppose that φ is an isomorphism. From Theorem 3.11 we get $n \in \mathbb{Z}_{>0}$ and a unital equivariant homomorphism $\psi_M: E_{d(M)} \rightarrow C/J_n$ such that $\pi_n \circ \psi_M = \varphi \circ \iota_{\infty, M}$. Applying Lemma 4.15 to $\pi = \varphi^{-1} \circ \pi_n: C/J_n \rightarrow \mathcal{O}_\infty$, for $m \geq M$ we inductively construct unital equivariant homomorphisms $\psi_m: E_{d(m)} \rightarrow C/J_n$ such that

$$\pi \circ \psi_{m+1} = \varphi \circ \iota_{\infty, m+1} \quad \text{and} \quad \psi_{m+1} \circ \iota_{m+1, m-1} = \psi_m \circ \iota_{m, m-1}.$$

Then $r_j = \lim_{m \rightarrow \infty} \psi_m(r_j^{d(m)})$ exists for all $g \in G$ and $j \in \mathbb{Z}_{>0}$, because when $d(m) \geq j$ it is equal to $\psi_{m+2}(r_j^{d(m+2)})$. So there is a unital equivariant homomorphism $\psi: \mathcal{O}_\infty \rightarrow C/J_n$ such that $\psi(s_j) = r_j$ for all $j \in \mathbb{Z}_{>0}$. Clearly $\pi \circ \psi = \varphi$.

For the general case, set $Q = \varphi(\mathcal{O}_\infty) \subset C/J$, let $D \subset C$ be the inverse image of Q , set $I_n = D \cap J_n$ for $n \in \mathbb{Z}_{>0}$, and set $I = D \cap J$. Then $I = \overline{\bigcup_{n=1}^\infty I_n}$. (In [20], see Proposition 13.1.4, Lemma 13.1.5, and the discussion afterwards.) So $Q = D/I$. Since \mathcal{O}_∞ is simple, the corestriction $\varphi_0: \mathcal{O}_\infty \rightarrow D/I$ of φ is an isomorphism. The result follows by applying the special case above with D in place of C , with I_n in place of J_n , and with φ_0 in place of φ . \square

Problem 4.17. Let G be an infinite compact group. Is a quasifree action of G on \mathcal{O}_∞ necessarily equivariantly semiprojective?

As a test case, consider the quasifree action coming from the left regular representation of S^1 .

5. EQUIVARIANTLY STABLE RELATIONS

We relate equivariant semiprojectivity to equivariant stability of relations because, in the applications we have in mind [25], equivariant stability of relations is what we actually use.

Weak stability of relations (Definition 4.1.1 of [20]) also has an equivariant version. Since equivariant stability holds for the examples we care about, we only consider equivariant stability.

We follow Section 13.2 of [20] for our definition of generators and relations.

For reference, we give the version of the definition without the group action, except that we give a version for unital C^* -algebras. This is a variant of Definition 13.2.1 of [20].

Definition 5.1. Let S be a set. We denote by F_S the universal unital C^* -algebra generated by the elements of S subject to the relations $\|s\| \leq 2$ for all $s \in S$. A *set of relations* on S is a subset $R \subset F_S$. We refer to (S, R) as a *set of generators and relations*. We say that (S, R) is *finite* if S and R are finite. We define $I_R \subset F_S$ to be the ideal in F_S generated by R .

Since we are asking for unital algebras and homomorphisms, we make the following definition.

Definition 5.2. A set (S, R) of generators and relations as in Definition 5.1 is *admissible* if $I_R \neq F_S$. When (S, R) is admissible, we let $\tau_R: F_S \rightarrow F_S/I_R$ be the quotient map. The *C^* -algebra on the generators and relations (S, R)* , which we write $C^*(S, R)$, is by definition F_S/I_R . We say that (S, R) is *bounded* if for every $s \in S$, we have $\|\tau_R(s)\| \leq 1$.

The choices $\|s\| \leq 2$ and $\|\tau_R(s)\| \leq 1$ are convenient normalizations. By scaling, every set of generators and relations can be fit in this framework.

The following is essentially Definition 13.2.2 of [20], but for the unital situation. By convention, we declare (except in a few places where we explicitly allow it) that the zero C^* -algebra is not unital.

Definition 5.3. Let the notation be as in Definition 5.1, let A be a unital C^* -algebra, and let $\rho: S \rightarrow A$ be a function such that $\|\rho(s)\| \leq 2$ for all $s \in S$. In

this situation, we write $\varphi^\rho: F_S \rightarrow A$ for the corresponding homomorphism. We say that ρ is a *representation of (S, R) in A* if $\varphi^\rho(x) = 0$ for all $x \in R$. For $\delta \in [0, 1]$, we say that ρ is a δ -*representation of (S, R) in A* if $\|\varphi^\rho(x)\| \leq \delta$ for all $x \in R$. (Sometimes, we will also allow the map to the zero C^* -algebra as a representation.) If (S, R) is admissible, then the *universal representation* ρ_R is obtained by taking $A = C^*(S, R)$ and $\rho_R = \tau_R|_S$.

Remark 5.4. It is clear that the universal representation, as defined above, really has the appropriate universal property.

Lemma 5.5. Let (S, R) be a set of generators and relations as in Definition 5.1. Then (S, R) is admissible if and only if there exists a representation in a (nonzero) unital C^* -algebra.

Proof. This is immediate. \square

Remark 5.6. We make some general remarks.

- (1) The relation corresponding to an element $x \in F_S$ is really just the statement $x = 0$. Here x could be any $*$ -polynomial in the noncommuting variables S , but in fact we are allowing arbitrary elements of the C^* -algebra F_S . The framework we describe in fact allows much more general relations. For example, suppose $R_0 \subset F_S$, $M: R_0 \rightarrow [0, \infty)$ is a function, and we want the relations to say $\|x\| \leq M(x)$ for all $x \in R_0$. We simply take the intersection $I \subset F_S$ of the kernels of all unital homomorphisms $\varphi: F_S \rightarrow A$, for arbitrary unital C^* -algebras A , such that $\|\varphi(x)\| \leq M(x)$ for all $x \in R_0$. Then we take as relations all elements of I , that is, we take $R = I$.

Positivity conditions on elements of F_S can be handled the same way.

- (2) If S is countable, we may always take R to be finite. Choose a countable subset $\{x_1, x_2, \dots\}$ of the unit ball of I_R whose span is dense in I_R . Then we can take the relations to consist of the single element

$$a = \sum_{n=1}^{\infty} 2^{-n} x_n^* x_n.$$

(This change does, however, change the meaning of a δ -representation.)

- (3) It follows from (1) and (2) that if (S, R) is finite and bounded, and $\delta \in [0, 1]$, then the universal C^* -algebra generated by a δ -representation of (R, S) is again the universal C^* -algebra on a finite and bounded set of generators and relations.
- (4) We have made a choice in the definition of a δ -representation: we still require $\|\rho(s)\| \leq 1$ for all $s \in S$. By suitable scaling and application of (1) above, it is also possible to get a version in which we merely require $\|\rho(s)\| \leq 1 + \delta$ for all $s \in S$.

We now give equivariant versions of these definitions. We restrict to discrete groups, and to finite groups in practice. If G is not discrete, but the universal C^* -algebra is supposed to carry a continuous action of G , then the relations must demand that the action of G on each generator defines a continuous function from G to the universal C^* -algebra. There are many kinds of conditions on elements of a C^* -algebra which can be made into relations which determine a universal C^* -algebra, but continuity of functions from the set of generators isn't one of them. The universal algebra will in general only be an inverse limit of C^* -algebras. See

Definition 1.3.4 and Proposition 1.3.6 of [23]. There do exist examples of universal G -algebras on generators and relations when G is not discrete. See Example 5.18 and Example 5.19 below. However, we leave the development of the appropriate theory for elsewhere.

Notation 5.7. Let S be a set, let G be a discrete group, and let σ be an action of G on S , written $(g, s) \mapsto \sigma_g(s)$. We denote by μ^σ the action of G on F_S induced by σ .

Definition 5.8. Let G be a discrete group. A G -equivariant set of generators and relations is a triple (S, σ, R) in which (S, R) is a set of generators and relations as in Definition 5.1, σ is an action of G on S (just as a set), and R is invariant under the action μ^σ of Notation 5.7. We say that (S, σ, R) is *admissible* if (S, R) is admissible in the sense of Definition 5.2. We say that (S, σ, R) is *bounded* if (S, R) is, and is *finite* if G and (S, R) are finite.

It may seem better to omit σ and the requirement of G -invariance, and to allow the group action in the relations. We address this formulation starting with Definition 5.13 below. However, doing so does not give anything new, and the version we have given above is technically more convenient.

Definition 5.9. Let G be a discrete group. Let (S, σ, R) be a G -equivariant set of generators and relations in the sense of Definition 5.8. Let $\alpha: G \rightarrow \text{Aut}(A)$ be an action of G on a unital C^* -algebra A . An *equivariant representation of (S, σ, R) in A* is a representation of (S, R) in the sense of Definition 5.3 such that for every $g \in G$ and $s \in S$, we have $\rho(\sigma_g(s)) = \alpha_g(\rho(s))$. For $\delta_1, \delta_2 \in [0, 1]$, a δ_1 -equivariant δ_2 -representation of (S, σ, R) in A is a δ_2 -representation ρ of (S, R) such that $\|\rho(\sigma_g(s)) - \alpha_g(\rho(s))\| \leq \delta_1$ for all $g \in G$ and $s \in S$. When $\delta_1 = 0$, we speak of an *equivariant δ_2 -representation of (S, σ, R) in A* .

If (S, R) is admissible, then the *universal equivariant representation* ρ_R is obtained by taking $A = C^*(S, R)$, with the action $\bar{\mu}^\sigma: G \rightarrow \text{Aut}(C^*(S, R))$ coming from the fact that I_R is an invariant ideal for $\mu^\sigma: G \rightarrow \text{Aut}(F_S)$, and taking $\rho_R = \tau_R|_S$. We write $C^*(S, \sigma, R)$ for the algebra equipped with this action.

We show that we have the right definition of admissibility.

Lemma 5.10. Let G be a discrete group, and let (S, σ, R) be a G -equivariant set of generators and relations. Then (S, σ, R) is admissible if and only if there exists an equivariant representation in a (nonzero) unital G -algebra.

Proof. If there is an equivariant representation, then Lemma 5.5 implies that (S, R) is admissible, so that (S, σ, R) is admissible.

For the reverse, since $I_R \neq F_S$, the universal equivariant representation of Definition 5.9 is an equivariant representation in a unital G -algebra. \square

The universal equivariant representation, as in Definition 5.9, really is universal.

Lemma 5.11. Let G be a discrete group, and let (S, σ, R) be a G -equivariant set of generators and relations. Let $\alpha: G \rightarrow \text{Aut}(A)$ be an action of G on a unital C^* -algebra A , and let $\rho: S \rightarrow A$ be an equivariant representation of (S, σ, R) in A . Then there exists a unique equivariant homomorphism $\varphi: C^*(S, \sigma, R) \rightarrow A$ such that $\varphi \circ \rho_R = \rho$.

Proof. As an algebra, we have $C^*(S, \sigma, R) = C^*(S, R)$. So Remark 5.4 provides a unique homomorphism $\varphi: C^*(S, \sigma, R) \rightarrow A$ such that $\varphi \circ \rho_R = \rho$. Let $\overline{\mu}^\sigma: G \rightarrow \text{Aut}(C^*(S, \sigma, R))$ be as in Definition 5.9. Since ρ is equivariant, for all $g \in G$ and $s \in S$ we have

$$\varphi(\overline{\mu}_g^\sigma(\tau_R(s))) = \varphi(\tau_R(\mu_g^\sigma(s))) = \rho(\sigma_g(s)) = \alpha_g(\rho(s)) = \alpha_g(\varphi(\tau_R(s))).$$

Since $\tau_R(S)$ generates $C^*(S, \sigma, R)$, equivariance of φ follows. \square

We have equivariant analogs of the first two parts of Remark 5.6.

Remark 5.12. (1) Let G be a discrete group, let S be a set, and let σ be an action of G on S . For any proper G -invariant ideal $I \subset F_S$, we can get F_S/I as a universal G -algebra $C^*(S, \sigma, R)$ simply by taking $R = I$.

As an example, let $R \subset F_S$ be G -invariant, and let $M: R \rightarrow [0, \infty)$ be a function such that $M(\sigma_g(s)) = M(s)$ for all $g \in G$ and $s \in S$. We take $I \subset F_S$ to be the intersection of the kernels of all unital equivariant homomorphisms $\varphi: F_S \rightarrow A$, for arbitrary unital G -algebras (G, A, α) , such that $\|\varphi(x)\| \leq M(x)$ for all $x \in R$.

- (2) If S is countable and G is finite, we always take R to be finite. Choose a countable subset $\{x_1, x_2, \dots\}$ of the unit ball of I_R whose span is dense in I_R . Then we can take the relations to consist of the single G -invariant element

$$a = \sum_{n=1}^{\infty} \sum_{g \in G} 2^{-n} \mu_g^\sigma(x_n^* x_n).$$

- (3) If (S, σ, R) is finite and bounded, and $\delta \in [0, 1)$, then the universal C^* -algebra generated by an equivariant δ -representation of (S, σ, R) is again the universal C^* -algebra on a finite and bounded set of generators and relations. However, for $\delta_0 > 0$, there is no obvious action of G on the universal C^* -algebra generated by a δ_0 -equivariant δ -representation of (S, σ, R) .

If we want to allow the action of G to appear in the relations, we can use the following alternate definition. We omit the word “equivariant” in the name.

Definition 5.13. Let G be a discrete group. A *set of generators and relations for a G -algebra* is a pair (S, R) in which S is a set and R is a subset of $F_{G \times S}$. Define an action σ of G on $G \times S$ by $\sigma_g(h, s) = (gh, s)$ for $g, h \in G$ and $s \in S$, and let $\mu^\sigma: G \rightarrow \text{Aut}(F_{G \times S})$ be as in Notation 5.7. The *associated G -equivariant set of generators and relations* to (S, R) is then

$$\left(G \times S, \sigma, \bigcup_{g \in G} \mu_g^\sigma(R) \right).$$

We let $I_{G,R} \subset F_{G \times S}$ be the ideal generated by $\bigcup_{g \in G} \mu_g^\sigma(R)$. We say that (S, R) is *admissible* if $I_{G,R} \neq F_{G \times S}$, and in this case we define the *universal G -algebra generated by (S, R)* to be $C^*(S, R) = F_{G \times S} / I_{G,R}$, with the action $\overline{\mu}: G \rightarrow \text{Aut}(C^*(S, R))$ induced by the action $\mu^\sigma: G \rightarrow \text{Aut}(F_{G \times S})$. Let $\tau_{G,R}: F_{G \times S} \rightarrow C^*(S, R)$ be quotient map. We say that (S, R) is *bounded* if for every $s \in S$, we have $\|\tau_{G,R}(1, s)\| \leq 1$. We say that (S, R) is *finite* if G , S , and R are all finite.

Definition 5.14. Let G be a discrete group, and let (S, R) be a set of generators and relations for a G -algebra in the sense of Definition 5.13. Let $\alpha: G \rightarrow \text{Aut}(A)$ be an action of G on a unital C^* -algebra A . A *representation of (S, R) in A* is a function

$\rho: S \rightarrow A$ such that the function $\pi: G \times S \rightarrow A$, defined by $\pi(g, s) = \alpha_g(\rho(s))$ for $g \in G$ and $s \in S$, is an equivariant representation, in the sense of Definition 5.9, of the associated G -equivariant set of generators and relations. For $\delta \in [0, 1]$, we say that ρ is a δ -representation of (S, R) in A if, using the notation of Definition 5.3, we have $\|\varphi^\pi(x)\| \leq \delta$ for all $x \in R$.

Remark 5.15. Let G be a discrete group, let (S, R) be a set of generators and relations for a G -algebra in the sense of Definition 5.13, and let the notation be as there. Set $Q = \bigcup_{g \in G} \mu_g^\sigma(R)$. Then:

- (1) (S, R) is admissible if and only if $(G \times S, \sigma, Q)$ is admissible in the sense of Definition 5.8.
- (2) (S, R) is bounded if and only if $(G \times S, \sigma, Q)$ is bounded in the sense of Definition 5.8. (Use the fact that for $g \in G$ and $s \in S$, we have $\tau_{G,R}(g, s) = \overline{\mu}_g(\tau_{G,R}(1, s))$.)
- (3) (S, R) is finite if and only if $(G \times S, \sigma, Q)$ is finite in the sense of Definition 5.8.
- (4) There is a unique equivariant isomorphism $\psi: C^*(S, R) \rightarrow C^*(G \times S, \sigma, Q)$ such that $\psi(\tau_{G,R}(g, s)) = \tau_R(g, s)$ for all $g \in G$ and $s \in S$.

We then get the following universal property for $C^*(S, R)$. The proof is clear, and is omitted.

Lemma 5.16. Let G be a discrete group, and let (S, R) be a set of generators and relations for a G -algebra in the sense of Definition 5.13. Let $\alpha: G \rightarrow \text{Aut}(A)$ be an action of G on a unital C^* -algebra A , and let $\rho: S \rightarrow A$ be a representation of (S, R) in A . Then there exists a unique equivariant homomorphism $\varphi: C^*(S, R) \rightarrow A$ such that $\varphi \circ \rho_R = \rho$.

Remark 5.17. Analogously to Remark 5.6(1) and Remark 5.12(1), we can now speak of the universal G -algebra generated by a set S with relations given by norm bounds and positivity conditions on $*$ -polynomials in the noncommuting variables $\bigsqcup_{g \in G} \sigma_g(S)$, that is, polynomials in the noncommuting variables consisting of the generators, their formal adjoints, and the formal images of all these under an action of G .

We now present examples to show that there are some cases in which there is a reasonable universal C^* -algebra, with continuous action of G , even with G not discrete.

Example 5.18. Let G be any topological group, and let (G, A, α) be any G -algebra. Take the generating set S to be the closed unit ball of A , take R to be the collection of all algebraic relations that hold among elements of S and their adjoints, and take $\sigma = \alpha|_S$. Then the universal C^* -algebra generated by (S, σ, R) is just A , with the representation being the identity map and the action of G being α . The algebra A is universal when G is given the discrete topology, but the action is in fact continuous when G is given its original topology.

One can make a slightly more interesting example as follows.

Example 5.19. Take $A = M_n$, and take α to be any action of G on M_n . Let $(e_{j,k})_{j,k=1}^n$ be the standard system of matrix units in M_n . Take the generators to consist of elements $v_{g,j,k}$ for $g \in G$ and $j, k \in \{1, 2, \dots, n\}$. Set $\sigma_g(v_{h,j,k}) = v_{gh,j,k}$.

The universal representation is intended to be $\rho(v_{g,j,k}) = \alpha_g(e_{j,k})$. To make this happen, take the relations to say that for each $g \in G$, the collection $(v_{g,j,k})_{j,k=1}^n$ is a system of matrix units, and also to include, for all $g, h \in G$ and $j, k \in \{1, 2, \dots, n\}$, the relation corresponding to the (unique) expression of $\alpha_g(\alpha_h(e_{j,k}))$ as a linear combination of the matrix units $\alpha_h(e_{l,m})$.

The following is the equivariant analog of Definition 14.1.1 of [20]. Following [20], we restrict to finite sets of generators and relations. Accordingly, we take the group to be finite.

Definition 5.20. Let G be a finite group, and let (S, σ, R) be a finite admissible G -equivariant set of generators and relations. Then we say that (S, σ, R) is *stable* if for every $\varepsilon > 0$ there is $\delta > 0$ such that the following holds. Suppose that (G, A, α) and (G, B, β) are unital G -algebras (except that we allow $B = 0$), that $\omega: A \rightarrow B$ is an equivariant homomorphism, that $\rho_0: S \rightarrow A$ is a δ -equivariant δ -representation of (S, σ, R) (in the sense of Definition 5.9), and that $\omega \circ \rho_0$ is an equivariant representation of (S, σ, R) . Then there exists an equivariant representation $\rho: S \rightarrow A$ of (S, σ, R) such that $\omega \circ \rho = \omega \circ \rho_0$ and such that for all $s \in S$ we have $\|\rho(s) - \rho_0(s)\| < \varepsilon$.

We allow $B = 0$ to incorporate the possibility that we are merely given a δ -equivariant δ -representation of (S, σ, R) but no homomorphism ω such that $\omega \circ \rho_0$ is an equivariant representation.

Lemma 5.21. Let G be a finite group, and let (S, σ, R) be a bounded finite admissible G -equivariant set of generators and relations. Then for every $\eta > 0$ there is $\delta > 0$ such that whenever (G, A, α) and (G, B, β) are unital G -algebras (with possibly $B = 0$), $\omega: A \rightarrow B$ is equivariant, and $\rho_0: S \rightarrow A$ is a δ -equivariant δ -representation of (S, σ, R) such that $\omega \circ \rho_0$ is an equivariant representation of (S, σ, R) , then there exists an (exactly) equivariant η -representation $\rho: S \rightarrow A$ such that $\omega \circ \rho = \omega \circ \rho_0$ and $\|\rho(s) - \rho_0(s)\| < \eta$ for all $s \in S$.

Proof. Since S and R are finite, there is $\delta_0 > 0$ such that whenever C is a C^* -algebra and $\psi_1, \psi_2: F_S \rightarrow C$ are two unital homomorphisms such that $\|\psi_1(s) - \psi_2(s)\| < \delta_0$ for all $s \in S$, then $\|\psi_1(r) - \psi_2(r)\| < \frac{1}{2}\eta$ for all $r \in R$. Set $\delta = \min(\delta_0, \frac{1}{2}\eta)$.

Now let ρ_0 be as in the hypotheses. For $s \in S$, define

$$\rho(s) = \frac{1}{\text{card}(G)} \sum_{g \in G} (\alpha_g \circ \rho_0 \circ \sigma_g^{-1})(s).$$

Then ρ is exactly equivariant. Also, for all $s \in S$, we have

$$\|\rho(s)\| \leq \max\{\|\rho_0(t)\| : t \in S\} \leq 2$$

and, since ρ_0 is δ -equivariant, $\|\rho(s) - \rho_0(s)\| \leq \delta \leq \delta_0$. Therefore, in the notation of Definition 5.3, for all $r \in R$ we have $\|\varphi^\rho(r) - \varphi^{\rho_0}(r)\| \leq \frac{1}{2}\eta$, whence

$$\|\varphi^\rho(r)\| \leq \frac{1}{2}\eta + \|\varphi^{\rho_0}(r)\| \leq \frac{1}{2}\eta + \delta \leq \eta.$$

Thus ρ is an η -representation. From

$$\omega \circ \alpha_g \circ \rho_0 \circ \sigma_g^{-1} = \beta_g \circ \omega \circ \rho_0 \circ \sigma_g^{-1} = \omega \circ \rho_0,$$

we get $\omega \circ \rho = \omega \circ \rho_0$, completing the proof. \square

Theorem 5.22. Let G be a finite group, and let (S, σ, R) be a bounded finite admissible G -equivariant set of generators and relations. Then (S, σ, R) is stable if and only if $C^*(S, \sigma, R)$ is equivariantly semiprojective.

Proof. Proposition 13.2.5 of [20] holds equally well, and with the same proof, for unital algebras, for a bounded finite admissible G -equivariant set (S, σ, R) of generators and relations (with, in particular, G finite), for an equivariant direct system of unital G -algebras with unital maps, and for a δ -equivariant δ -representation of (S, σ, R) . Therefore stability of (S, σ, R) implies equivariant semiprojectivity of $C^*(S, \sigma, R)$.

The proof of the reverse implication roughly follows the proof for the nonequivariant case, as, for example, in the proof of Theorem 14.1.4 of [20]. For $n \in \mathbb{Z}_{>0}$ let $J_n \subset F_S$ be the intersection of the kernels of the homomorphisms φ^ρ as ρ runs through all equivariant 2^{-n} -representations of (S, σ, R) . Then J_n is a G -invariant ideal in F_S ,

$$J_1 \subset J_2 \subset \cdots, \quad \text{and} \quad \overline{\bigcup_{n=1}^{\infty} J_n} = I_R.$$

The quotient F_S/J_n is the universal G -algebra generated by an equivariant 2^{-n} -representation of (S, σ, R) . We will apply the definition of equivariant semiprojectivity to $C^*(S, \sigma, R)$, with $C = F_S$, with J_n as given, with $J = I_R$, and with $\varphi = \text{id}_{C^*(S, \sigma, R)}$. We use the same names $\kappa: F_S \rightarrow F_S/I_R$, $\kappa_n: F_S \rightarrow F_S/J_n$, $\pi_n: F_S/J_n \rightarrow F_S/I_R$, etc. for the maps as in Definition 1.1 and Remark 1.3(3). By equivariant semiprojectivity, we can choose $n_0 \in \mathbb{Z}_{>0}$ and a unital equivariant homomorphism $\psi_0: C^*(S, \sigma, R) \rightarrow F_S/J_{n_0}$ such that $\pi_{n_0} \circ \psi_0 = \text{id}_{C^*(S, \sigma, R)}$.

For $s \in S$ we have

$$(\pi_{n_0} \circ \psi_0 \circ \kappa)(s) = (\pi_{n_0} \circ \psi_0 \circ \pi_{n_0} \circ \kappa_{n_0})(s) = (\pi_{n_0} \circ \kappa_{n_0})(s).$$

Since S is finite, there is $n \geq n_0$ such that for all $s \in S$ we have

$$\|(\pi_{n, n_0} \circ \psi_0 \circ \kappa)(s) - \kappa_n(s)\| = \|(\pi_{n, n_0} \circ \psi_0 \circ \kappa)(s) - (\pi_{n, n_0} \circ \kappa_{n_0})(s)\| < \frac{1}{2}\varepsilon.$$

We may also require that $2^{-n} < \frac{1}{2}\varepsilon$. Define $\psi = \pi_{n, n_0} \circ \psi_0$, getting $\pi_n \circ \psi = \text{id}_{C^*(S, \sigma, R)}$ and

$$(5.1) \quad \|(\psi \circ \kappa)(s) - \kappa_n(s)\| < \frac{1}{2}\varepsilon$$

for all $s \in S$.

Choose $\delta > 0$ as in Lemma 5.21 for $\eta = 2^{-n}$. Let (G, A, α) and (G, B, β) be unital G -algebras (with possibly $B = 0$), let $\omega: A \rightarrow B$ be equivariant, and let $\rho_0: S \rightarrow A$ be a δ -equivariant δ -representation of (S, σ, R) such that $\omega \circ \rho_0$ is an equivariant representation of (S, σ, R) . By the choice of δ , there is an equivariant 2^{-n} -representation $\rho_1: S \rightarrow A$ such that $\omega \circ \rho_1 = \omega \circ \rho_0$ and

$$(5.2) \quad \|\rho_1(s) - \rho_0(s)\| < 2^{-n}$$

for all $s \in S$.

The following diagram (in which the triangle and the square will be shown to commute, and we already know that $\pi_n \circ \kappa_n = \kappa$) shows some of the maps we have

or which will be constructed:

$$\begin{array}{ccccccc}
 & & & \xrightarrow{\quad \kappa \quad} & & & \\
 & & \swarrow & & \searrow & & \\
 S & \xrightarrow{\quad} & F_S & \xrightarrow{\quad \kappa_n \quad} & F_S/J_n & \xrightarrow{\quad \pi_n \quad} & F_S/I_R \\
 & \searrow & & & \downarrow \varphi & & \downarrow \lambda \\
 & & & & A & \xrightarrow{\quad \omega \quad} & B
 \end{array}$$

By the definition of J_n , there is a unital equivariant homomorphism $\varphi: F_S/J_n \rightarrow A$ such that $\varphi(\kappa_n(s)) = \rho_1(s)$ for all $s \in S$. Define $\rho(s) = (\varphi \circ \psi \circ \kappa)(s)$ for $s \in S$. Then ρ is an equivariant representation of (S, σ, R) . Moreover, there is an equivariant homomorphism $\lambda: F_S/I_R \rightarrow B$ such that $\lambda(\kappa(s)) = \omega(\rho_1(s))$ for all $s \in S$. By construction, for $s \in S$ we have

$$(\omega \circ \varphi \circ \kappa_n)(s) = (\omega \circ \rho_1)(s) = (\lambda \circ \pi_n \circ \kappa_n)(s).$$

Since κ_n is surjective and S generates F_S , we get $\omega \circ \varphi = \lambda \circ \pi_n$. For $s \in S$ we now have

$$\begin{aligned}
 (\omega \circ \rho)(s) &= (\omega \circ \varphi \circ \psi \circ \kappa)(s) = (\lambda \circ \pi_n \circ \psi \circ \kappa)(s) \\
 &= (\lambda \circ \kappa)(s) = (\omega \circ \rho_1)(s) = (\omega \circ \rho_0)(s).
 \end{aligned}$$

That is, $\omega \circ \rho = \omega \circ \rho_0$.

It remains only to show that $\|\rho(s) - \rho_0(s)\| < \varepsilon$ for $s \in S$. Using (5.2) and $2^{-n} < \frac{1}{2}\varepsilon$ at the second step, we have

$$\|\rho(s) - \rho_0(s)\| \leq \|\rho(s) - \rho_1(s)\| + \|\rho_1(s) - \rho_0(s)\| < \|\rho(s) - \rho_1(s)\| + \frac{1}{2}\varepsilon,$$

and, by (5.1),

$$\|\rho(s) - \rho_1(s)\| = \|(\varphi \circ \psi \circ \kappa)(s) - (\varphi \circ \kappa_n)(s)\| \leq \|(\psi \circ \kappa)(s) - \kappa_n(s)\| < \frac{1}{2}\varepsilon.$$

The required estimate follows, and the theorem is proved. \square

We now consider the version of stability in which the group action is allowed in the relations.

Definition 5.23. Let G be a finite group, and let (S, R) be a finite admissible set of generators and relations for a G -algebra, in the sense of Definition 5.13. We say that (S, R) is *stable* if for every $\varepsilon > 0$ there is $\delta > 0$ such that the following holds. Suppose that (G, A, α) and (G, B, β) are unital G -algebras (except that we allow $B = 0$), that $\omega: A \rightarrow B$ is an equivariant homomorphism, that $\rho_0: S \rightarrow A$ is a δ -representation of (S, R) (in the sense of Definition 5.14), and that $\omega \circ \rho_0$ is a representation of (S, R) . Then there exists a representation $\rho: S \rightarrow A$ of (S, R) such that $\omega \circ \rho = \omega \circ \rho_0$ and such that for all $s \in S$ we have $\|\rho(s) - \rho_0(s)\| < \varepsilon$.

Lemma 5.24. Let G be a finite group, and let (S, R) be a finite bounded admissible set of generators and relations for a G -algebra in the sense of Definition 5.13. Let the action σ of G on $G \times S$ be as there, and set $Q = \bigcup_{g \in G} \mu_g^\sigma(R)$, so that the associated G -equivariant set of generators and relations is $(G \times S, \sigma, Q)$. Then:

- (1) For every $\eta > 0$ there is $\delta > 0$ such that whenever (G, A, α) is a unital G -algebra and $\lambda: G \times S \rightarrow A$ is a δ -equivariant δ -representation of $(G \times S, \sigma, Q)$, then the function $s \mapsto \lambda(1, s)$ is an η -representation of (S, R) .

- (2) For every $\eta > 0$ there is $\delta > 0$ such that whenever (G, A, α) is a unital G -algebra and $\rho: S \rightarrow A$ is a δ -representation of (S, R) , then the function $(g, s) \mapsto \alpha_g(\rho(s))$ is an equivariant η -representation of $(G \times S, \sigma, Q)$.

Proof. We prove part (1). Suppose the conclusion fails. Apply Definition 5.14 and use finiteness of R to find $x \in R$, $\eta > 0$, and for each $n \in \mathbb{Z}_{>0}$ a unital G -algebra $(G, A_n, \alpha^{(n)})$ and a $\frac{1}{n}$ -equivariant $\frac{1}{n}$ -representation $\lambda_n: G \times S \rightarrow A_n$ such that, if we define $\rho_n(s) = \lambda_n(1, s)$ for $s \in S$ and $\pi_n(g, s) = \alpha_g^{(n)}(\rho_n(s))$ for $g \in G$ and $s \in S$, then, following the notation of Definition 5.3, we have $\|\varphi^{\pi_n}(x)\| > \eta$.

Let $\prod_{n=1}^{\infty} A_n$ be the C^* -algebraic product (the set of sequences $(a_n)_{n \in \mathbb{Z}_{>0}}$ in the algebraic product such that $\sup_{n \in \mathbb{Z}_{>0}} \|a_n\|$ is finite), and define

$$A = \prod_{n=1}^{\infty} A_n \Big/ \bigoplus_{n=1}^{\infty} A_n.$$

The obvious coordinatewise definitions, followed by the quotient map, give an action $\alpha: G \rightarrow \text{Aut}(A)$ and functions

$$\lambda: G \times S \rightarrow A, \quad \rho: S \rightarrow A, \quad \text{and} \quad \pi: G \times S \rightarrow A.$$

One checks that λ is an equivariant representation of $(G \times S, \sigma, Q)$. Clearly $\rho(s) = \lambda(1, s)$ for $s \in S$ and $\pi(g, s) = \alpha_g(\rho(s))$ for $g \in G$ and $s \in S$. Therefore $\pi = \lambda$. Since $x \in Q$, we have $\varphi^{\pi}(x) = 0$. This contradicts the fact that $\|\varphi^{\pi_n}(x)\| > \eta$ for all $n \in \mathbb{Z}_{>0}$. Part (1) is proved.

Now suppose part (2) is false. Since Q is finite, there exist $x \in Q$, $\eta > 0$, and for each $n \in \mathbb{Z}_{>0}$ a unital G -algebra $(G, A_n, \alpha^{(n)})$ and a $\frac{1}{n}$ -representation $\rho_n: S \rightarrow A_n$ such that, if we define $\pi_n(g, s) = \alpha_g^{(n)}(\rho_n(s))$ for $g \in G$ and $s \in S$, then $\|\varphi^{\pi_n}(x)\| > \eta$. The functions π_n are equivariant. Define A , α , ρ , and π as in the proof of part (1). Then ρ is a representation of (S, R) , π is an equivariant representation of $(G \times S, \sigma, Q)$, and $\pi(g, s) = \alpha_g(\rho(s))$ for all $g \in G$ and $s \in S$. Therefore $\varphi^{\pi}(x) = 0$, contradicting $\|\varphi^{\pi_n}(x)\| > \eta$ for all $n \in \mathbb{Z}_{>0}$. \square

Theorem 5.25. Let G be a finite group, and let (S, R) be a bounded finite admissible set of generators and relations for a G -algebra. Then $C^*(S, R)$ is equivariantly semiprojective if and only if (S, R) is stable in the sense of Definition 5.23.

Proof. Define σ , μ , and Q as in Lemma 5.24. It follows from Remark 5.15 that $(G \times S, \sigma, Q)$ is bounded, finite, and admissible. By Theorem 5.22 and Remark 5.15(4), it therefore suffices to prove that (S, R) is stable if and only if $(G \times S, \sigma, Q)$ is stable in the sense of Definition 5.20.

Assume that (S, R) is stable. Let $\varepsilon > 0$. Choose $\eta > 0$ as in Definition 5.23 (where the number is called δ), for $\frac{1}{2}\varepsilon$ in place of ε . Choose $\delta_0 > 0$ following Lemma 5.24(1) (where the number is called δ). We may also require that $\delta_0 \leq \frac{1}{2}\varepsilon$. Apply Lemma 5.21, with δ_0 in place of η , to get a number $\delta > 0$.

Let (G, A, α) and (G, B, β) be unital G -algebras (except that we allow $B = 0$), and let $\omega: A \rightarrow B$ be an equivariant homomorphism. Let $\lambda_0: G \times S \rightarrow A$ be a δ -equivariant δ -representation of $(G \times S, \sigma, Q)$ such that $\omega \circ \lambda_0$ is an equivariant representation of (S, σ, R) . By the choice of δ , there is an equivariant δ_0 -representation $\lambda_1: G \times S \rightarrow A$ such that $\omega \circ \lambda_1 = \omega \circ \lambda_0$ and $\|\lambda_1(g, s) - \lambda_0(g, s)\| < \delta_0$ for all $g \in G$ and $s \in S$.

Define $\rho_1: S \rightarrow A$ by $\rho_1(s) = \lambda_1(1, s)$ for $s \in S$. Since λ_1 is equivariant, ρ_1 is a δ_0 -representation of (S, R) . Clearly $\omega \circ \rho_1$ is a representation of (S, R) . By the

choice of δ_0 , there exists a representation $\rho: S \rightarrow A$ of (S, R) such that $\omega \circ \rho = \omega \circ \rho_1$ and such that for all $s \in S$ we have $\|\rho(s) - \rho_1(s)\| < \frac{1}{2}\varepsilon$. Define $\lambda: G \times S \rightarrow A$ by $\lambda(g, s) = \alpha_g(\rho(s))$ for $g \in G$ and $s \in S$. Then, using equivariance of ω at the first step, we have $\omega \circ \lambda = \omega \circ \lambda_1 = \omega \circ \lambda_0$. Moreover, for $g \in G$ and $s \in S$, by equivariance of λ and λ_1 , we have $\lambda(g, s) = \alpha_g(\rho(s))$ and $\lambda_1(g, s) = \alpha_g(\rho_1(s))$. Therefore

$$\|\lambda(g, s) - \lambda_0(g, s)\| \leq \|\alpha_g(\rho(s)) - \alpha_g(\rho_1(s))\| + \|\lambda_1(g, s) - \lambda_0(g, s)\| < \frac{1}{2}\varepsilon + \delta_0 \leq \varepsilon.$$

This completes the proof that $(G \times S, \sigma, Q)$ is stable.

For the reverse, assume that $(G \times S, \sigma, Q)$ is stable. We prove that (S, R) is stable. Let $\varepsilon > 0$. Choose $\eta > 0$ as in Definition 5.20 (where the number is called δ). Choose $\delta > 0$ as in Lemma 5.24(2). Let (G, A, α) and (G, B, β) be unital G -algebras (except that we allow $B = 0$), and let $\omega: A \rightarrow B$ be an equivariant homomorphism. Let $\rho_0: S \rightarrow A$ be a δ -representation of (S, R) such that $\omega \circ \rho_0$ is a representation of (S, R) . Define $\pi_0: G \times S \rightarrow A$ by $\pi_0(g, s) = \alpha_g(\rho_0(s))$ for $g \in G$ and $s \in S$. Then π_0 is an equivariant δ -representation of $(G \times S, \sigma, Q)$. Therefore there exists an equivariant representation $\pi: G \times S \rightarrow A$ of (S, σ, R) such that $\omega \circ \pi = \omega \circ \pi_0$ and such that for all $g \in G$ and $s \in S$ we have $\|\pi(g, s) - \pi_0(g, s)\| < \varepsilon$. Define $\rho: S \rightarrow A$ by $\rho(s) = \pi(1, s)$ for $s \in S$. Then $\|\rho(s) - \rho_0(s)\| < \varepsilon$ for all $s \in S$. Also clearly $\omega \circ \rho = \omega \circ \rho_0$. This completes the proof of the theorem. \square

As an immediate application, we can derive stronger versions of the Rokhlin property for actions of finite groups (Definition 3.1 of [14]; formulated without the central sequence algebra in Definition 1.1 of [24]) and the tracial Rokhlin property (Definition 1.2 of [24]).

Proposition 5.26. Let A be a separable unital C^* -algebra, and let $\alpha: G \rightarrow \text{Aut}(A)$ be an action of a finite group G on A . Then α has the Rokhlin property if and only if for every finite set $F \subset A$ and every $\varepsilon > 0$, there are mutually orthogonal projections $e_g \in A$ for $g \in G$ such that:

- (1) $\alpha_g(e_h) = e_{gh}$ for all $g, h \in G$.
- (2) $\|e_g a - a e_g\| < \varepsilon$ for all $g \in G$ and all $a \in F$.
- (3) $\sum_{g \in G} e_g = 1$.

The definition of the Rokhlin property differs in that in condition (1), one merely requires $\|\alpha_g(e_h) - e_{gh}\| < \varepsilon$ for all $g, h \in G$.

The proof is very similar to, but simpler than, the proof of Proposition 5.27, and is omitted.

Proposition 5.27. Let A be an infinite dimensional simple separable unital C^* -algebra, and let $\alpha: G \rightarrow \text{Aut}(A)$ be an action of a finite group G on A . Then α has the tracial Rokhlin property if and only if for every finite set $F \subset A$, every $\varepsilon > 0$, and every positive element $x \in A$ with $\|x\| = 1$, there are mutually orthogonal projections $e_g \in A$ for $g \in G$ such that:

- (1) $\alpha_g(e_h) = e_{gh}$ for all $g, h \in G$.
- (2) $\|e_g a - a e_g\| < \varepsilon$ for all $g \in G$ and all $a \in F$.
- (3) With $e = \sum_{g \in G} e_g$, the projection $1 - e$ is Murray-von Neumann equivalent to a projection in the hereditary subalgebra of A generated by x .
- (4) With e as in (3), we have $\|e x e\| > 1 - \varepsilon$.

The definition of the tracial Rokhlin property differs in that in condition (1) one merely requires $\|\alpha_g(e_h) - e_{gh}\| < \varepsilon$ for all $g, h \in G$.

We give the details of the proof to demonstrate how our machinery works, and in particular to show why we do not want to require our δ -representations to be exactly equivariant.

Proof of Proposition 5.27. Let $F \subset A$ be finite and let $\varepsilon > 0$. By scaling, without loss of generality $\|a\| \leq 1$ for all $a \in F$. Set $n = \text{card}(G)$ and

$$\varepsilon_0 = \min\left(\frac{1}{n}, \frac{\varepsilon}{2n+1}\right).$$

Let S consist of distinct elements p_g for $g \in G$. Define an action σ of G on S by $\sigma_g(p_h) = p_{gh}$ for $g, h \in G$. Define

$$R = \{p_g p_h - \delta_{g,h} p_g : g, h \in G\} \cup \{p_g^* - p_g : g \in G\}.$$

Then (S, σ, R) is an equivariant set of generators and relations, and $C^*(S, \sigma, R)$ is equivariantly isomorphic to $C(G) \oplus \mathbb{C}$, with the action on $C(G)$ coming from the translation action of G on itself and the trivial action on \mathbb{C} , in such a way that p_g is sent to $(\chi_{\{g\}}, 0)$. This action is equivariantly semiprojective by Theorem 2.6. So (S, σ, R) is stable by Theorem 5.22. Choose $\delta_0 > 0$ as in the definition of stability for ε_0 in place of ε . Set $\delta = \min(\delta_0, \varepsilon_0)$. Apply the tracial Rokhlin property with δ in place of ε , obtaining mutually orthogonal projections $e_g^{(0)} \in A$ for $g \in G$ such that:

- (5) $\|\alpha_g(e_h^{(0)}) - e_{gh}^{(0)}\| < \delta$ for all $g, h \in G$.
- (6) $\|e_g^{(0)} a - a e_g^{(0)}\| < \delta$ for all $g \in G$ and all $a \in F$.
- (7) With $e^{(0)} = \sum_{g \in G} e_g^{(0)}$, the projection $1 - e^{(0)}$ is Murray-von Neumann equivalent to a projection in the hereditary subalgebra of A generated by x .
- (8) With $e^{(0)}$ as in (7), we have $\|e^{(0)} x e^{(0)}\| > 1 - \varepsilon_0$.

Define $\rho_0 : S \rightarrow A$ by $\rho_0(p_g) = e_g^{(0)}$ for $g \in G$. Then ρ_0 is a δ -equivariant δ -representation of (S, σ, R) . Therefore there is an equivariant representation ρ of (S, σ, R) such that $\|\rho(p_g) - e_g^{(0)}\| < \varepsilon_0$ for all $g \in G$. Set $e_g = \rho(p_g)$ for $g \in G$. By the definition of an equivariant representation, the e_g are mutually orthogonal projections satisfying condition (1). Condition (2) follows from the estimates

$$\|a\| \leq 1, \quad \|e_g - e_g^{(0)}\| < \varepsilon_0 \leq \frac{1}{3}\varepsilon, \quad \text{and} \quad \|e_g^{(0)} a - a e_g^{(0)}\| < \delta \leq \varepsilon_0 \leq \frac{1}{3}\varepsilon$$

for $g \in G$ and $a \in F$.

It remains to prove conditions (3) and (4). Set $e = \sum_{g \in G} e_g$. First, we have

$$\|e - e^{(0)}\| < n\varepsilon_0.$$

Since $n\varepsilon_0 \leq 1$, the projection e is Murray-von Neumann equivalent to $e^{(0)}$, and therefore also to a projection in the hereditary subalgebra of A generated by x . This is (3). Moreover,

$$\|e x e\| \geq \|e^{(0)} x e^{(0)}\| - 2\|e - e^{(0)}\| > 1 - \varepsilon_0 - 2n\varepsilon_0 \geq 1 - \varepsilon.$$

This is (4). □

6. GRADED SEMIPROJECTIVITY OF THE C*-ALGEBRA OF A FINITE GROUP

In this section, we show that if G is a finite group then $C^*(G)$, with its natural G -grading, is semiprojective in the graded sense. This is an application of Lemma 2.4, the same result that played a key role in the proof that finite dimensional C*-algebras are equivariantly semiprojective.

Presumably much more general results are possible. Indeed, the appropriate setting may be actions of finite dimensional Hopf algebras or compact quantum groups on finite dimensional C*-algebras.

The following definition is a special case of Definitions 3.1 and 3.4 of [10], of a C*-algebra (topologically) graded by a discrete group G . In [10], the group is not necessarily finite, and one only requires that $\bigoplus_{g \in G} A_g$ be dense in A . Continuity of the projection to A_1 (as in Definition 3.4 of [10]) is automatic when the group is finite and $\bigoplus_{g \in G} A_g = A$.

Definition 6.1. Let G be a finite group, and let A be a C*-algebra. A G -grading on A is a direct sum decomposition as Banach spaces

$$A = \bigoplus_{g \in G} A_g$$

such that if $g, h \in G$, $a \in A_g$, and $b \in A_h$, then $ab \in A_{gh}$ and $a^* \in A_{g^{-1}}$. (We do not say anything about the direct sum norm except that it is equivalent to the usual norm on A .)

A subspace $E \subset A$ is *graded* if $E = \sum_{g \in G} (E \cap A_g)$.

We denote by P_g , or P_g^A , the projection map from A to A_g associated with this direct sum decomposition.

To put this definition in context, we make three remarks. First, when G is finite, a G -grading of A is the same as an identification of A with the C*-algebra of a Fell bundle over G . The basic correspondence is given in VIII.16.11 and VIII.16.12 of [11], but in general it is not bijective. It is bijective for Fell bundles over discrete groups which are amenable in the sense of Definition 4.1 of [10], and in particular for all Fell bundles and topological gradings when G is amenable. (This follows from Theorem 4.7 of [10].) Since our groups are finite, the correspondence is bijective in our case.

Second, for discrete groups G , a normal coaction on a C*-algebra A (as defined before Definition 1.1 of [28]) is the same as an identification of A with the C*-algebra of a Fell bundle over G . See Proposition 3.3 and Theorem 3.8 of [28].

Finally, if G is abelian, then a G -grading on A is the same as an action $\alpha: \widehat{G} \rightarrow \text{Aut}(A)$. Given a G -grading on A and $\tau \in \widehat{G}$, we define $\alpha_\tau \in \text{Aut}(A)$ by $\alpha_\tau(a) = \tau(g)a$ for $a \in A_g$. Given α , for $g \in G$ we set

$$A_g = \{a \in A: \alpha_\tau(a) = \tau(g)a \text{ for all } \tau \in \widehat{G}\},$$

that is, A_g is the spectral subspace for g when g is regarded as an element of the second dual of G .

Remark 6.2. Let G be a finite group, and let $A = \bigoplus_{g \in G} A_g$ be a G -grading of A . Then the summand A_1 is a C*-algebra. (This is clear.) Let $P_g: A \rightarrow A_g$ be as in Definition 6.1. Then P_1 is a conditional expectation onto A_1 , and $\|P_g\| \leq 1$ for all $g \in G$. (See Theorem 3.3 and Corollary 3.5 of [10].)

Definition 6.3. Let G be a finite group, let A and B be C^* -algebras with G -gradings $A = \bigoplus_{g \in G} A_g$ and $B = \bigoplus_{g \in G} B_g$, and let $\varphi: A \rightarrow B$ be a homomorphism. We say that φ is *graded* if for every $g \in G$ we have $\varphi(A_g) \subset B_g$.

Remark 6.4. Let G be a finite group, let A be a C^* -algebra with G -grading $A = \bigoplus_{g \in G} A_g$, and let $I \subset A$ be a graded ideal. Then A/I becomes a graded C^* -algebra with the grading

$$(A/I)_g = A_g / (A_g \cap I) = (A_g + I) / I,$$

and the quotient map $A \rightarrow A/I$ is a graded homomorphism.

Remark 6.5. Let G be a finite group. Then the direct limit of a direct system of G -graded C^* -algebras with graded maps is a G -graded C^* -algebra in an obvious way.

Remark 6.6. Let G be a finite group, let A be a C^* -algebra, and let $\alpha: G \rightarrow \text{Aut}(A)$ be an action of G on A . Then the crossed product $C^*(G, A, \alpha)$ is graded in the following way. Let $u_g \in C^*(G, A, \alpha)$ (or in $M(C^*(G, A, \alpha))$ if A is not unital) be the standard unitary corresponding to $g \in G$. Then

$$C^*(G, A, \alpha)_g = \{au_g : a \in A\}.$$

(This is the dual coaction.)

Remark 6.7. In Remark 6.6, take $A = \mathbb{C}$ and take α to be the trivial action. This gives a canonical G -grading on $C^*(G)$. If $u_g \in C^*(G)$ is the unitary corresponding to $g \in G$, then $C^*(G)_g = \mathbb{C}u_g$.

The following definition is the analog of Definition 14.1.3 of [20].

Definition 6.8. Let G be a finite group, and let A be a C^* -algebra with G -grading $A = \bigoplus_{g \in G} A_g$. We say that the grading is *graded semiprojective* if whenever C is a C^* -algebra with G -grading $C = \bigoplus_{g \in G} C_g$, $J_0 \subset J_1 \subset \dots$ are graded ideals in C , $J = \overline{\bigcup_{n=0}^{\infty} J_n}$, and $\varphi: A \rightarrow C/J$ is a graded homomorphism, then there exists n and a graded homomorphism $\psi: A \rightarrow C/J_n$ such that the composition

$$A \xrightarrow{\psi} C/J_n \longrightarrow C/J$$

is equal to φ .

When no confusion can arise, we say that A is graded semiprojective.

Here is the diagram:

$$\begin{array}{ccc} & C & \\ & \downarrow & \\ & C/J_n & \\ \nearrow \psi & \downarrow & \\ A & \xrightarrow{\varphi} & C/J \end{array}$$

Theorem 6.9. Let G be a finite group. Then $C^*(G)$, with the G -grading in Remark 6.7, is graded semiprojective.

Proof. Set $\varepsilon_0 = \frac{1}{6 \cdot 34}$, and choose $\varepsilon > 0$ such that $\varepsilon \leq \varepsilon_0$ and such that whenever A is a unital C^* -algebra, $u \in U(A)$, and $a \in A$ satisfies $\|a - u\| < \varepsilon$, then we have $\|a(a^*a)^{-1/2} - u\| < \varepsilon_0$.

Let the notation be as in Definition 6.8 and Remark 1.3(3). Further, for $g \in G$ let $P_g^{(n)}: C/J_n \rightarrow (C/J_n)_g$ and $P_g: C/J \rightarrow (C/J)_g$ be the projection maps associated to the gradings, as in Definition 6.1. Also let $u_g \in C^*(G)$ be the unitary associated with the group element $g \in G$, as in Remark 6.7.

Let $\varphi: A \rightarrow C/J$ be a unital graded homomorphism. Since finite dimensional C^* -algebras are semiprojective, there exist n_0 and a unital homomorphism (not necessarily graded) $\psi_0: A \rightarrow C/J_{n_0}$ which lifts φ . Since π_{n_0} and φ are graded, for $g \in G$ we have

$$\pi_{n_0}(\psi_0(u_g) - P_g^{(n_0)}(\psi_0(u_g))) = \varphi(u_g) - P_g(\varphi(u_g)) = 0.$$

Therefore there exists $n \geq n_0$ such that for all $g \in G$ we have

$$(6.1) \quad \|\pi_{n,n_0}(\psi_0(u_g) - P_g^{(n_0)}(\psi_0(u_g)))\| < \varepsilon.$$

Set $\psi_1 = \pi_{n,n_0} \circ \psi_0$ and for $g \in G$ set $c_g = P_g^{(n)}(\psi_1(u_g))$, which is in $(C/J)_g$. Then (6.1) becomes

$$\|\psi_1(u_g) - c_g\| < \varepsilon$$

for all $g \in G$.

Since $\varepsilon < 1$, we can define $\rho_0: G \rightarrow U(C/J_n)$ by $\rho_0(g) = c_g(c_g^*c_g)^{-1/2}$. Since $c_g \in (C/J)_g$, we have $c_g^*c_g \in (C/J)_1$, whence $(c_g^*c_g)^{-1/2} \in (C/J)_1$, so that $\rho_0(g) \in (C/J)_g$. Moreover, the choice of ε ensures that $\|c_g - \rho_0(g)\| < \varepsilon_0$. Therefore

$$\|\rho_0(g) - \psi_1(u_g)\| \leq \|\rho_0(g) - c_g\| + \|c_g - \psi_1(u_g)\| < \varepsilon_0 + \varepsilon \leq 2\varepsilon_0.$$

Let $g, h \in G$. Since

$$\psi_1(u_g), \psi_1(u_h) \in U(C/J_n) \quad \text{and} \quad \psi_1(u_{gh}) = \psi_1(u_g)\psi_1(u_h),$$

it follows that

$$\|\rho_0(gh) - \rho_0(g)\rho_0(h)\| < 6\varepsilon_0.$$

Since $\pi_n(c_g) = \varphi(u_g)$ is unitary, we also get

$$\pi_n(\rho_0(g)) = \pi_n(c_g) = \varphi(u_g).$$

Inductively define functions $\rho_m: G \rightarrow U(C/J_n)$ by (following Lemma 2.4)

$$\rho_{m+1}(g) = \exp\left(\frac{1}{\text{card}(G)} \sum_{h \in G} \log\left(\rho_m(h)^* \rho_m(hg) \rho_m(g)^*\right)\right) \rho_m(g)$$

for $g \in G$. Since $6\varepsilon_0 = \frac{1}{34} < \frac{1}{17}$, Lemma 2.4 implies that the functions ρ_m are well defined maps $\rho_m: G \rightarrow U(C/J_n)$ such that $\rho(g) = \lim_{m \rightarrow \infty} \rho_m(g)$ defines a homomorphism $\rho: G \rightarrow U(C/J_n)$ satisfying

$$\|\rho(g) - \rho_0(g)\| \leq \frac{2 \cdot 6\varepsilon_0}{1 - 17 \cdot 6\varepsilon_0} \quad \text{and} \quad \pi_n(\rho(g)) = \varphi(u_g)$$

for all $g \in G$.

We claim that $\rho_m(g) \in (C/J_n)_g$ for all $g \in G$ and $m \in \mathbb{Z}_{\geq 0}$. The proof is by induction on m . We know this is true for $m = 0$. Assume it is true for m . For all $g, h \in G$ we have

$$\rho_m(h)^* \rho_m(hg) \rho_m(g)^* \in (C/J_n)_h^* (C/J_n)_{hg} (C/J_n)_g^* \subset (C/J_n)_1.$$

Therefore also

$$\exp \left(\frac{1}{\text{card}(G)} \sum_{h \in G} \log \left(\rho_m(h)^* \rho_m(hg) \rho_m(g)^* \right) \right) \in (C/J_n)_1,$$

and the induction step follows. This proves the claim. Taking limits, we get $\rho(g) \in (C/J_n)_g$ for all $g \in G$.

By the universal property of $C^*(G)$, there is a unital homomorphism $\psi: C^*(G) \rightarrow C/J$ such that $\psi(u_g) = \rho(g)$ for all $g \in G$. By construction, ψ is graded. Moreover, $\pi_n \circ \psi(u_g) = \varphi(u_g)$ for all $g \in G$, so the universal property of $C^*(G)$ implies that $\pi_n \circ \psi = \varphi$. Thus ψ lifts φ . \square

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DEPARTMENT OF MATHEMATICS, UNIVERSITY OF OREGON, EUGENE OR 97403-1222, USA,
 AND RESEARCH INSTITUTE FOR MATHEMATICAL SCIENCES, KYOTO UNIVERSITY, KITASHIRAKAWA-
 OIWAKECHO, SAKYO-KU, KYOTO 606-8502, JAPAN.

E-mail address: ncp@darkwing.uoregon.edu